MAT 21C: PRACTICE PROBLEMS LECTURE 6

PROFESSOR CASALS (SECTIONS B01-08)

ABSTRACT. Practice problems for the sixth lecture of Part II, delivered May 12 2023. Solutions will be posted within 48h of these problems being posted.

Problem 1. Consider the vectors $v = \langle -2, 1, -1\rangle$ and $u = \langle 4, -5, 7\rangle$.

(a) Compute the cross product $u \times v$.
Recall for vectors $u = \langle u_1, u_2, u_3\rangle, v = \langle v_1, v_2, v_3\rangle$, we have that

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} i - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} j + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} k \tag{1}$$

Substitute $\langle 4, -5, 7\rangle, \langle -2, 1, -1\rangle$ for u, v respectively.

$$u \times v = \begin{vmatrix} -5 & 7 \\ 1 & -1 \end{vmatrix} i - \begin{vmatrix} 4 & 7 \\ -2 & -1 \end{vmatrix} j + \begin{vmatrix} 4 & -5 \\ -2 & 1 \end{vmatrix} k = -2i - 10j - 6 = \langle -2, -10, -6\rangle.$$

(b) By explicit computation and only for this example, verify that the cross product $u \times v$ is orthogonal to both u and v.
To show $u \times v$ is orthogonal to u and v, it suffices to show that $(u \times v) \cdot u$ (dot product) and $(u \times v) \cdot v$ both equal 0. We compute the dot products:

$$(u \times v) \cdot u = \langle -2, -10, -6\rangle \cdot \langle 4, -5, 7\rangle = (-2) \cdot 4 + (-10) \cdot (-5) + (-6) \cdot 7 = 0$$
$$(u \times v) \cdot v = \langle -2, -10, -6\rangle \cdot \langle -2, 1, -1\rangle = (-2) \cdot (-2) + (-10) \cdot 1 + (-6) \cdot (-1) = 0$$

So $u \times v$ is indeed orthogonal to both u and v.

(c) Find the area of the parallelogram spanned by u and v.
The area of the parallelogram spanned by u and v is determined by $|u \times v|$. This is precisely: $|u \times v| = (-2)^2 + (-10)^2 + (-6)^2 = 140$.

Problem 2. Consider the vectors $v = \langle 3, -7, 1\rangle$ and $w = \langle -1, 4, 2\rangle$.

(a) Show that the cross product is $v \times w = \langle -18, -7, 5\rangle$.
By Equation 1, we have:

$$v \times w = \begin{vmatrix} -7 & 1 \\ 4 & 2 \end{vmatrix} i - \begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix} j + \begin{vmatrix} 3 & -7 \\ -1 & 4 \end{vmatrix} k = \langle -18, -7, 5\rangle$$

(b) Find an equation for the plane π which contains both u and v and passes through the origin $(0, 0, 0)$.
Recall that for any plane, all we need is a point on the plane and a normal vector. One such point is $(0, 0, 0)$. From the construction of π in the proof of PSet 5 Question 6, the equation of π is:

$$\pi : \langle -18, 7, 5\rangle \cdot \langle x - 0, y - 0, z - 0\rangle = -18x - 7y + 5z = 0$$

Problem 3. Consider the parallelogram with vertices $(0, 0, 0), (4, 5, -11), (-3, 2, 17), (1, 7, 6)$.

(a) Show that the parallelogram is spanned by $u = \langle 4, 5, -11\rangle$ and $v = \langle -3, 2, 17\rangle$.
We want to show that every point on the parallelogram P is a linear combination of u and v. Consider $s, t \in [0, 1]$. Then we have the following:
- Any point on the line between $(0, 0, 0)$ and $(4, 5, -11)$ can be determined by su.
- Any point on the line between $(0, 0, 0)$ and $(-3, 2, 17)$ can be determined by tv.
- Any point on the line between $(-3, 2, 17)$ and $(1, 7, 6)$ can be determined by $v + su$
- Any point on the line between $(4, 5, -11)$ and $(1, 7, 6)$ can be determined by $u + tv$.

(b) Find the area of the parallelogram.
In the previous part we showed that P is spanned by u, v. Then the area of P is given by $|u \times v|$. We compute:

$$u \times v = \begin{vmatrix} 5 & -11 \\ 2 & 17 \end{vmatrix} i - \begin{vmatrix} 4 & -11 \\ -3 & 17 \end{vmatrix} j + \begin{vmatrix} 4 & 5 \\ -3 & 2 \end{vmatrix} k = \langle 107, -35, 23\rangle$$

Then we have that $|u \times v| = \sqrt{107^2 + (-35)^2 + 23^2}$.

Problem 4. Let u, v be two vectors.

(a) Show by direct computation that $u \times v = -v \times u$.
Firstly:

$$u \times v = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} i - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} j + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} k$$

Secondly, we have:

$$v \times u = \begin{vmatrix} v_2 & v_3 \\ u_2 & u_3 \end{vmatrix} i - \begin{vmatrix} v_1 & v_3 \\ u_1 & u_3 \end{vmatrix} j + \begin{vmatrix} v_1 & v_2 \\ u_1 & u_2 \end{vmatrix} k$$

Notice the i component of $u \times v$ is $u_2 v_3 - u_3 v_2 = -(v_2 u_3 - u_2 v_3)$. We see this is also true for the j, k components as well. Thus $u \times v = -v \times u$.

(b) Argue geometrically that $u \times v = -v \times u$.
Consider u, v; v has a specific orientation with respect to u. This orientation is the opposite when referring to u with respect to v (For example in 2D: If v is some clockwise rotation from u, then u must be a counter-clockwise rotation of the same angle to v). The orientation of u and v with respect to one another matters when talking about $u \times v$ and $v \times u$. To preserve the orientation of u and v when considering $u \times v$, we must negate one of the vectors, in this case v, in order to preserve the direction of the output vector $-v \times u$.

Problem 5. Suppose that $u = \langle u_1, u_2, u_3 \rangle, v = \langle v_1, v_2, v_3 \rangle$ are vectors.

(a) Prove by direct computation that u, v are parallel, i.e. we have equality of the ratios $\frac{u_1}{v_1} = \frac{u_2}{v_2} = \frac{u_3}{v_3}$, if and only if $u \times v$=0.

Proof. ($\implies$) Suppose that u and v are parallel, in other words, there exists $c \in \mathbb{R}$ such that $u = cv$. Then we want to show $u \times v = 0$. By Equation 1, we have:

$$\begin{aligned} u \times v &= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} i - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} j + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} k \\ &= \begin{vmatrix} cv_2 & cv_3 \\ v_2 & v_3 \end{vmatrix} i - \begin{vmatrix} cv_1 & cv_3 \\ v_1 & v_3 \end{vmatrix} j + \begin{vmatrix} cv_1 & cv_2 \\ v_1 & v_2 \end{vmatrix} k \\ &= 0i + 0j = 0k = 0. \end{aligned}$$

($\impliedby$) Now suppose $u \times v = 0$. Then we have that each:

$$\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \quad \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \quad \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

are equal 0. This is true if and only if $u_i v_j - u_j v_i = 0 \implies u_i v_j = u_j v_i \implies \frac{u_i}{v_i} = \frac{u_j}{v_j}$ for $1 \leq i, j \leq 3$. Then we conclude that u, v must be parallel. ∎

(b) Using the cross product angle formula show that two vectors u, v parallel if and only if $u \times v = 0$.

Proof. Suppose u, v are parallel. This is true if and only if the angle θ between u and v is 0. We have

$$|u||v| \sin \theta = 0. \tag{2}$$

The only vector with magnitude 0 is the zero vector (or if $u, v = 0$, in which case the statement is true by convention), and because $|u \times v| = |u||v| \sin \theta$, Equation 2 is true if and only if $u \times v = 0$. ∎

(c) Justify geometrically that u, v parallel if and only if $u \times v = 0$.
Recall that $|u \times v|$ is the area of the parallelogram spanned by vectors u and v. If u and v are parallel, then the parallelogram spanned by these vectors is a line, which has area 0.

Problem 6. Using the cross product, find an equation for the unique plane π containing the points $(0, 0, 0)$, $(2, -5, -8)$ and $(11, -7, 34)$.

We will again use the construction of a plane from PSet 5 Question 6. Two vectors on π are $u = \langle 2, -5, -8\rangle$ and $v = \langle 11, -7, 34\rangle$. We need a normal vector and a point to construct the plane, so compute $u \times v$ and we get this is $\langle -226, -156, 41\rangle$. One point on π is $(0, 0, 0)$

$$\pi : -226(x - 0) - 156(y + 0) + 41(z + 0) = -226x - 156y + 41z = 0$$