MAT 21C: PRACTICE PROBLEMS LECTURE 8

PROFESSOR CASALS (SECTIONS B01-08)

Abstract. Practice problems for the eighth lecture of Part II, delivered May 17 2023. Solutions will be posted within 48h of these problems being posted.

Problem 1. Consider the unique plane π containing the three points $P = (1, 0, 2), Q = (-2, 3, 0)$ and $R = (0, -5, 1)$.

(a) Find the distance from the point $S = (1, 2, -4)$ to π using the vector $\vec{PS}$. Recall that the distance from a point $\mathbf{S}$ to a plane through some point $\mathbf{X}$ with a normal $\mathbf{n}$ is given by $d = \left|\vec{XS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right|$.

We will take the cross product of $\vec{PQ}$ and $\vec{PR}$ to find the normal vector. Hence, $\mathbf{n} = \langle -3, 3, -2\rangle \times \langle -1, -5, -1\rangle = \langle -13, -1, 18\rangle$ and $|\mathbf{n}| = \sqrt{494}$.

Use the vector $\vec{PS} = \langle 0, 2, -6\rangle$.

Thus, $d = \left|\langle 0, 2, -6\rangle \cdot \langle \frac{-13}{\sqrt{494}}, \frac{-1}{\sqrt{494}}, \frac{18}{\sqrt{494}}\rangle\right| = \left|\frac{-2}{\sqrt{494}} - \frac{108}{\sqrt{494}}\right|$

(b) Find the distance from the point $S = (1, 2, -4)$ to π using the vector $\vec{QS}$.

Use the vector $\vec{QS} = \langle 3, -1, -4\rangle$.

Thus, $d = \left|\langle 3, -1, -4\rangle \cdot \langle \frac{-13}{\sqrt{494}}, \frac{-1}{\sqrt{494}}, \frac{18}{\sqrt{494}}\rangle\right| = \left|\frac{-39}{\sqrt{494}} + \frac{1}{\sqrt{494}} - \frac{72}{\sqrt{494}}\right|$

(c) Find the distance from the point $S = (1, 2, -4)$ to π using the vector $\vec{RS}$.

Use the vector $\vec{RS} = \langle 1, 7, -5\rangle$.

Thus, $d = \left|\langle 1, 7, -5\rangle \cdot \langle \frac{-13}{\sqrt{494}}, \frac{-1}{\sqrt{494}}, \frac{18}{\sqrt{494}}\rangle\right| = \left|\frac{-13}{\sqrt{494}} - \frac{7}{\sqrt{494}} - \frac{90}{\sqrt{494}}\right|$

Problem 2. Consider the three planes

$$\pi_1 = \{3x - 5y + 4z = 12\}$$
$$\pi_2 = \{\text{unique plane that contains } (0, 1, 0) \text{ with perpendicular direction } \langle 1, 4, 3\rangle\}$$
$$\pi_3 = \{\text{unique plane that contains } (0, 0, 0) \text{ and vectors } u = \langle 2, 4, 1\rangle, v = \langle 2, -5, 12\rangle\}$$

and the point $S = (-2, 0, 1)$.

(a) Find the distance of S to π_1.

If a plane is of the form $Ax + By + Cz = D$, then $\mathbf{n} = \langle A, B, C\rangle$. Thus, $\mathbf{n} = \langle 3, -5, 4\rangle$ and $|\mathbf{n}| = \sqrt{50}$. Take the point $X = (4, 0, 0)$ to get $\vec{XS} = \langle -6, 0, 1\rangle$. Thus, $d = \left|\vec{XS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = \left|\frac{-18}{\sqrt{50}} + \frac{4}{\sqrt{50}}\right|$

(b) Find the distance of S to π_2.

Here, $\mathbf{n} = \langle 1, 4, 3\rangle, |\mathbf{n}| = \sqrt{26}$, and $\vec{XS} = \langle -2, -1, 1\rangle$. Hence, $d = \left|\langle -2, -1, 1\rangle \cdot \langle \frac{1}{\sqrt{26}}, \frac{4}{\sqrt{26}}, \frac{3}{\sqrt{26}}\rangle\right| = \left|\frac{-2}{\sqrt{26}} - \frac{4}{\sqrt{26}} + \frac{3}{\sqrt{26}}\right|$

(c) Find the distance of S to π_3.

Here, $\mathbf{n} = \mathbf{u} \times \mathbf{v} = \langle 53, -22, 18\rangle, |\mathbf{n}| = \sqrt{3617}$, and $\vec{XS} = \langle -2, 0, 1\rangle$. Thus, $d = \left|\frac{-106}{\sqrt{3617}} + \frac{18}{\sqrt{3617}}\right|$

Problem 3. Find two different points S_1 and S_2 in space such that both S_1 and S_2 have distance to the plane $\{x + y + z = 0\}$ equal to 9.

Notice that the plane passes through the point (0,0,0) and has a normal $\mathbf{n} = \langle 1, 1, 1\rangle$, where $|\mathbf{n}| = \sqrt{3}$. We want to find two possible vectors $\vec{XS} = \langle a, b, c\rangle$ such that $\left|\langle a, b, c\rangle \cdot \frac{1}{\sqrt{3}}\langle 1, 1, 1\rangle\right| = 9 \implies |a+b+c| = 9\sqrt{3}$. We can say that $S_1 = (3\sqrt{3},\ 3\sqrt{3},\ 3\sqrt{3})$ and $S_2 = (0, 0,\ 9\sqrt{3})$.

Problem 4. Find two different planes π_1 and π_2 in space such that both π_1 and π_2 have distance to the point $S = (1, 0, 0)$ equal to 23.

Arbitrarily set both planes to have normal $\mathbf{n} = \langle 1, 0, 0\rangle$. We want to find two points that each respective plane passes through to satisfy the prompt above. Notice that with our choice of normal vectors, the planes π_1 and π_2 are parallel with the yz-plane, so this problem reduces to a one-dimensional problem. Any planes with $\mathbf{n} = \langle 1, 0, 0\rangle$ and pass through the point $x = 24$ or $x = -22$ satisfy our conditions.

More precisely, if our planes pass through the point $X = (a, b, c)$, then $d = |\langle 1 - a, -b, -c\rangle \cdot \langle 1, 0, 0\rangle| = |1 - a| = 23 \implies 1 - a = 23$ or $a - 1 = 23$.

Problem 5. Consider the two planes

$$\pi_1 = \{x - z = 12\}$$

$$\pi_2 = \{\text{unique plane that contains } (0, 0, 0) \text{ with perpendicular direction } \langle 1, 1, 1\rangle\}$$

(a) Compute the distance from $S = (11, 2, -4)$ to the plane π_1.

The plane π_1 contains the point $X = (12, 0, 0)$ and has normal line $\mathbf{n} = \langle 1, 0, -1\rangle$, so $d = \left|\langle -1, 2, -4\rangle \cdot \langle \frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\rangle\right| = \left|\frac{-1}{\sqrt{2}} + \frac{4}{\sqrt{2}}\right|$

(b) Compute the distance from $S = (11, 2, -4)$ to the plane π_2.

$d = \left|\langle 11, 2, -4\rangle \cdot \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\rangle\right| = \left|\frac{11}{\sqrt{3}} + \frac{2}{\sqrt{3}} - \frac{4}{\sqrt{3}}\right|$

(c) Compute the distance from $S = (11, 2, -4)$ to the intersection line $\pi_1 \cap \pi_2$.

The distance from some point S to a line that passes through P parallel to $\mathbf{v}$ is $d = \frac{|\vec{PS} \times \mathbf{v}|}{|\mathbf{v}|}$. Recall that the intersection line is parallel to $n_1 \times n_2$ where n_1 and n_2 are the normals of π_1 and π_2 respectively, so $\mathbf{v} = n_1 \times n_2 = \langle 1, 0, -1\rangle \times \langle 1, 1, 1\rangle = \langle 1, -2, 1\rangle$. To obtain the point P we find any (x, y, z) that satisfy $x - z = 12$ and $x + z + y = -12$. Arbitrarily set $z = 0$. Then $P = (12, -12, 0)$. Thus, $d = \frac{|\langle -1, 14, -4\rangle \times \langle 1, -2, 1\rangle|}{|\langle 1, -2, 1\rangle|} = \frac{|\langle 6, -3, -12\rangle|}{|\langle 1, -2, 1\rangle|} = \frac{3\sqrt{21}}{\sqrt{6}}$

Problem 6. Let L be the unique line through the point $P = (1, 2, 0)$ and direction vector $v = \langle 0, 2, -7\rangle$. Compute the distance from the point $S = (-3, 0, 4)$ to the line L.

$d = \frac{|\langle -4, -2, 4\rangle \times \langle 0, 2, -7\rangle|}{|\langle 0, 2, -7\rangle|} = \frac{|\langle 6, -28, -8\rangle|}{|\langle 0, 2, -7\rangle|} = \frac{2\sqrt{221}}{\sqrt{53}}$

Problem 7. Let L be the unique line through the points $P = (1, 2, 0)$ and $Q = (7, -5, 6)$. Compute the distance from the point $S = (-3, 0, 4)$ to the line L.

Here, $v = \vec{PQ} = \langle 6, -7, 6\rangle$. Thus, $d = \frac{|\langle -4,-2,4\rangle \times \langle 6,-7,6\rangle|}{|\langle 6,-7,6\rangle|} = \frac{|\langle 16,48,40\rangle|}{|\langle 6,-7,6\rangle|} = \frac{8\sqrt{65}}{\sqrt{121}}$

Problem 8. Decide whether each of the following sentences is *true* or *false*.

(a) A point P belongs to a line L if and only if the distance from P to L is zero.
True. The distance is zero when $\vec{P_0P} \times \mathbf{v} = \vec{0}$. If a point P belongs to the line, then $\vec{P_0P}$ is parallel to $\mathbf{v}$ and $\vec{P_0P} \times \mathbf{v} = \vec{0}$ is guaranteed.

(b) A point P belongs to a plane π if and only if the distance from P to π is zero.
True. If a point P belongs to a plane, then $\vec{P_0P}$ is perpendicular to the plane's normal vector $\mathbf{n}$. Thus, $\vec{P_0P} \cdot \mathbf{v} = 0 \implies d = 0$.

(c) Given a point P, there exists a unique plane π whose distance to P is 1.
False. Consider the point $P(0, 0, 0)$. The planes $z = 1$ and $x = 1$ are one unit away from the origin, but are two different planes.

(d) Given a point P, there are infinitely many lines L whose distance to P is 14.
True. We can find some plane π whose distance to P is also 14, then we generate infinitely many lines within π where all the lines pass through the point nearest to P. For example, consider the point (0,0,14) and any line in the xy-plane that passes through the origin.

(e) If a point P belongs to a plane π_1 and L is a line of intersection between π_1 and a different (non-parallel) plane π_2. Then the distance from P to L is the same as the distance from P to π_2.
False. Consider π_1 as the plane $z = 0$ (the xy-plane), π_2 as the plane $0.001x+z = 0$ (a slight rotation of the xy-plane about the y-axis), and the point P(1000,0,0). The distance between P and π_2 is 1, but the distance between P and L is 1000.