University of California Davis | **Name (Print):** ____________
Calculus MAT 21C | **Student ID (Print):** ____________

Practice Midterm II Examination | **May 26 2023**
Time Limit: 50 Minutes

This examination document contains 10 pages, including this cover page, and 4 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (25 points) Consider the points $P = (1, 0, 0)$, $Q = (-2, 0, 3)$ and $R = (-5, 1, -1)$.

   (a) (5 points) Compute the vector $\vec{PQ} \times \vec{PR}$.

   We compute the vectors $\vec{PQ} = Q - P = \langle -3, 0, 3\rangle$ and $\vec{PR} = R - P = \langle -6, 1, -1\rangle$. To compute their cross product we calculate

$$\begin{aligned}\vec{PQ} \times \vec{PR} &= \begin{vmatrix} i & j & k \\ -3 & 0 & 3 \\ -6 & 1 & -1 \end{vmatrix} \\ &= \begin{vmatrix} 0 & 3 \\ 1 & -1 \end{vmatrix} i + \begin{vmatrix} -3 & 3 \\ -6 & -1 \end{vmatrix} j + \begin{vmatrix} -3 & 0 \\ -6 & 1 \end{vmatrix} k \\ &= \langle -3, -21, -3\rangle\end{aligned}$$

   (b) (5 points) Consider the unique plane π containing P, Q and R. Explain why

$$\{-3x - 21y - 3z = -3\}$$

   is an equation for π.

   The unique plane π contains the points P, Q, and R so if the three points satisfy the equation, then it is an equation for π. We see that

$$\begin{aligned}-3(1) - 21(0) - 3(0) &= -3 \\ -3(-2) - 21(0) - 3(3) &= -3 \\ -3(-5) - 21(1) - 3(-1) &= -3\end{aligned}$$

   so $\{-3x - 21y - 3z = -3\}$ is an equation for π.

(c) (5 points) Justify that $v = (21, 0, -21)$ is a direction of the line L of intersection of π with the plane $\Pi = \{x + z = 1\}$.

For planes Π and π with normal vectors $\vec{n_\Pi} = \langle 1, 0, 1\rangle$ and $\vec{n_\pi} = \langle -3, -21, -3\rangle$ will be a line in the direction $\vec{n_\pi} \times \vec{n_\Pi}$. Thus,

$$\begin{aligned}\vec{n_\pi} \times \vec{n_\Pi} &= \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ -3 & -21 & -3 \end{vmatrix} \\ &= \begin{vmatrix} 0 & 1 \\ -21 & -3 \end{vmatrix} i + \begin{vmatrix} 1 & 1 \\ -3 & -3 \end{vmatrix} j + \begin{vmatrix} 1 & 0 \\ -3 & -21 \end{vmatrix} k \\ &= \langle 21, 0, -21\rangle\end{aligned}$$

so the line of intersection L, is in direction $\langle 21, 0, -21\rangle$.

(d) (5 points) Find the distance from $S = (0, 0, 2)$ to line L.

We find a point on both lines by setting $z = 0$ for both planar equations and finding the point P = (1,0,0) on the line of intersection, L.
The distance from point S to L through point P parallel to vector $\mathbf{v}$ is given by $d = \frac{|\vec{PS} \times \mathbf{v}|}{|\mathbf{v}|}$.
We calculate $\vec{PS} = \langle -1, 0, 2\rangle$ and normal vector for plane π $\vec{n} = \langle -3, -21, -3\rangle$. Computing each term in the formula,

$$\begin{aligned}\vec{PS} \times \mathbf{v} &= \begin{vmatrix} i & j & k \\ -1 & 0 & 2 \\ 21 & 0 & -21 \end{vmatrix} \\ &= \begin{vmatrix} 0 & 2 \\ 0 & -21 \end{vmatrix} i + \begin{vmatrix} -1 & 2 \\ 21 & -21 \end{vmatrix} j + \begin{vmatrix} -1 & 0 \\ 21 & 0 \end{vmatrix} k \\ &= \langle 0, 21, 0\rangle \\ |\vec{PS} \times \mathbf{v}| &= \sqrt{0^2 + 21^2 + 0^2} = 21 \\ |\mathbf{v}| &= \sqrt{21^2 + 0^2 + (-21)^2} = \sqrt{882} = 21\sqrt{2}\end{aligned}$$

Thus,

$$\begin{aligned}d &= \frac{|\vec{PS} \times \mathbf{v}|}{|\mathbf{v}|} \\ &= \frac{21}{21\sqrt{2}} \\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

(e) (5 points) Find the distance from $S = (0, 0, 2)$ to the plane π.

The distance from a point S to the plane π with normal vector $\mathbf{n}$ is given by the formula $d = |\vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}|$. We will again use point $P = (1, 0, 0)$ to find vector $\vec{PS} = \langle -1, 0, 2\rangle$.

We use the equation for the plane π to get $\mathbf{n} = \langle -3, -21, -3\rangle$ and

$$|\mathbf{n}| = \sqrt{(-3)^2 + (-21)^2 + (-3)^2} = \sqrt{459} = 3\sqrt{51}.$$

Thus,

$$\begin{aligned}\frac{\mathbf{n}}{|\mathbf{n}|} &= \frac{1}{3\sqrt{51}}\langle -3, -21, -3\rangle \\ &= \frac{1}{\sqrt{51}}\langle -1, -7, -3\rangle.\end{aligned}$$

Then,

$$\begin{aligned}d &= |\vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}| \\ &= |\langle -1, 0, 2\rangle \cdot \frac{1}{\sqrt{51}}\langle -1, -7, -3\rangle| \\ &= \frac{1}{\sqrt{51}}|\langle -1, 0, 2\rangle \cdot \langle -1, -7, -3\rangle| \\ &= \frac{1}{\sqrt{51}}|((-1)(-3) + 0(-21) + 2(-3))| \\ &= \frac{1}{\sqrt{51}}\end{aligned}$$

2. (25 points) Consider the vectors $u = \langle 2, 0, -1\rangle$ and $v = \langle 3, 4, -5\rangle$.

(a) (5 points) Show that $\langle 4, 7, 8\rangle$ is perpendicular to both u and v.

To show two vectors are perpendicular, we must show that their dot product is 0.

$$\langle 4, 7, 8\rangle \cdot \langle 2, 0, -1\rangle = (4 * 2) + (7 * 0) + (8(-1)) = 0$$
$$\langle 4, 7, 8\rangle \cdot \langle 3, 4, -5\rangle = (4 * 3) + (7 * 4) + (8(-5)) = 0$$

Because both dot products are 0, both u and v are perpendicular to $\langle 4, 7, 8\rangle$.

(b) (5 points) Argue that u is not parallel to v.

If u and v are parallel, then their cross product will be 0.

$$u \times v = \begin{vmatrix} i & j & k \\ 2 & 0 & -1 \\ 3 & 4 & -5 \end{vmatrix} = \langle 4, 7, 8\rangle \neq 0$$

.

Because $u \times v \neq 0$, u and v cannot be parallel.

(c) (5 points) Compute $\sin\theta$, where θ is the angle between u and v.

To find $\sin(\theta)$, we use both definitions of the cross product. Using out calculation from (b) we know $u \times v = \langle 4, 7, 8\rangle$. We then use the alternative cross product definition $|u \times v| = |u||v| * \sin(\theta)$.

$$|u \times v| = \sqrt{4^2 + 7^2 + 8^2} = \sqrt{129}$$
$$|u| = \sqrt{2^2 + 0^2 + (-1)^2} = \sqrt{5}$$
$$|v| = \sqrt{3^2 + 4^2 + (-5)^2} = \sqrt{50}$$

Then,

$$|u \times v| = |u||v| * \sin(\theta)$$
$$\sqrt{129} = \sqrt{5}\sqrt{50}\,\sin(\theta)$$
$$\Rightarrow \sin(\theta) = \frac{\sqrt{129}}{\sqrt{5}\sqrt{50}}$$

(d) (5 points) Verify that the vector $w = \langle 1, 0, 2\rangle$ is perpendicular to u but w is *not* perpendicular to v.

To check if two vectors are perpendicular, we compute the dot product.

$$w \cdot u = \langle 1, 0, 2\rangle \cdot \langle 2, 0, -1\rangle = (1 * 2) + (0 * 0) + (2(-1)) = 0$$
$$\Rightarrow w \perp u$$
$$w \cdot v = \langle 1, 0, 2\rangle \cdot \langle 3, 4, -5\rangle = (1 * 3) + (0 * 4) + (2(-5)) = -7$$
$$\Rightarrow w \not\perp v$$

(e) (5 points) Find a vector that is perpendicular to v but *not* perpendicular to u.

We are looking for a vector y such that $v \cdot y = 0$.
Let $y = \langle y_1, y_2, y_3\rangle$. Then,

$$v \cdot y = \langle 3, 4, -5\rangle \cdot \langle y_1, y_2, y_3\rangle = 3y_1 + 4y_2 - 5y_3 = 0.$$

We can choose any y_1, y_2, and y_3 that satisfy the above equation. Choosing $y_1 = 5$ and $y_2 = 0$, then y_3 must be 3. Thus a perpendicular vector to v is $y = \langle 5, 0, 3\rangle$
To see that u and y are not perpendicular we compute $u \cdot y$,

$$u \cdot y = \langle 2, 0, -1\rangle \cdot \langle 5, 0, 3\rangle = (2 * 5) + (0 * 0) + (-1 * 3) = 7 \neq 0$$

so u and y are not perpendicular.

3. (25 points) Consider a particle moving with a trajectory $\vec{r}(t) = \langle \cos(3t), \sin(4t), t^3 \rangle$.

(a) (5 points) Where will the particle be at time $t = \pi$?

To find the particles position at $t = \pi$, we compute $\vec{r}(\pi)$.

$$\begin{aligned} \vec{r}(\pi) &= \langle \cos(3\pi), \sin(4\pi), (\pi)^3 \rangle \\ &= \langle -1, 0, \pi^3 \rangle \end{aligned}$$

(b) (5 points) Find the velocity vector $\vec{v}(t)$ of the particle.

To find $\vec{v}(t)$ from $\vec{r}(t)$ we compute $\frac{\mathrm{d}}{\mathrm{d}t}\vec{r}(t)$.

$$\begin{aligned} \vec{v}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\vec{r}(t) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\langle \cos(3t), \sin(4t), t^3 \rangle \\ &= \langle -3\sin(3t), 4\cos(4t), 3t^2 \rangle \end{aligned}$$

(c) (5 points) Compute the speed of the particle at time $t = \pi$.

Speed is the scalar component of the velocity vector. Thus, we must compute $|\vec{v}(\pi)|$.

$$\begin{aligned} \vec{v}(\pi) &= \langle -3\sin(3\pi), 4\cos(4\pi), 3\pi^2 \rangle = \langle 0, 4, 3\pi^2 \rangle \\ |\vec{v}(\pi)| &= \sqrt{0^2 + 4^2 + (3\pi^2)^2} = \sqrt{16 + 9\pi^4} \end{aligned}$$

.

Thus the speed at $t = \pi$ is equal to $\sqrt{16 + 9\pi^4}$.

(d) (5 points) Show that the acceleration at $t = \pi$ is given by

$$a(\pi) = \langle 9, 0, 6\pi \rangle.$$

To find $\vec{a}(t)$ from $\vec{v}(t)$ we compute $\frac{\mathrm{d}}{\mathrm{d}t}\vec{v}(t)$.

$$\begin{aligned}\vec{a}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\vec{v}(t) \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \\ &= \langle -3\sin(3t), 4\cos(4t), 3t^2 \rangle \\ &= \langle -9\cos(3t), -16\sin(4t), 6t \rangle\end{aligned}$$

Substituting $t = \pi$ gives

$$\begin{aligned}\vec{a}(t) &= \langle -3\sin(3\pi), -16\cos(4\pi), 6\pi \rangle \\ &= \langle 9, 0, 6\pi \rangle\end{aligned}$$

(e) (5 points) Will there ever be a positive time t where the particle will be at rest, i.e. have zero speed?

If the particle comes to rest, then $\vec{v}(t) = 0$ for all t. The third component, $3t^2$, can only be 0 if $t = 0$. But, at $t = 0$, the second component will be $4\cos(0) = 4 \neq 0$ so there is no value of t that will make each component of $\vec{v}(t)$ 0 simultaneously. Thus, the particle will never have speed zero.

4. (25 points) For each of the statements below, circle the **unique** correct answer. (You do *not* need to justify your answer.)

(a) (5 points) The intersection of the sphere $(x-2)^2 + y^2 + (z+1)^2 \leq 16$ with the plane $\pi = \{2x - 11y + 5z = -1\}$ is:

(1) Empty. (2) A circle. (3) A disk. (4) A half-space. (5) A line.

(3) The center of the sphere, (2,0,-1), satisfies the equation of the plane so the plane passes through the center of the solid sphere. Thus, the intersection will be a disk.

(b) (5 points) The intersection of the plane $\pi_1 = \{x + y + z = 1\}$ with the plane $\pi_2 = \{5x + 5y + 5z = 17\}$ is:

(1) Empty. (2) A circle. (3) A line. (4) A point. (5) Two points.

(1) The cross product of u and v gives $\langle 0,0,0\rangle = \vec{0}$ so the planes are parallel and thus there is no intersection.

(c) (5 points) The cross product of $u = \langle -3, 2, 4\rangle$ and $v = \langle 6, -4, -8\rangle$:

(1) $\langle 0,0,0\rangle$ (2) $\langle 1,0,0\rangle$ (3) $\langle 0,1,0\rangle$ (4) $\langle 0,0,1\rangle$ (5) $\langle 1,1,1\rangle$.

(1) $\langle 0,0,0\rangle$

$$\langle -3,2,4\rangle \times \langle 6,-4,-8\rangle = \begin{vmatrix} i & j & k \\ -3 & 2 & 4 \\ 6 & -4 & -8 \end{vmatrix} = \langle 0,0,0\rangle$$

(d) (5 points) The midpoint between $P = (0,6,4)$ and $Q = (8,2,-4)$ is:

(1) $\langle 4,4,0\rangle$ (2) $\langle 4,-2,-4\rangle$ (3) $\langle 8,-4,8\rangle$ (4) $\langle 0,-8,4\rangle$ (5) $\langle 2,0,-4\rangle$.

(1) $\langle 4,4,0\rangle$

$$\frac{1}{2}(P+Q) = \frac{1}{2}\langle (0+8),(6+2),(4+(-4))\rangle = \frac{1}{2}\langle 8,8,0\rangle = \langle 4,4,0\rangle$$

(e) (5 points) A particle with trajectory $r(t) = (e^t, t+3, 5t)$ has speed at $t = 0$:

(1) 0. (2) $\sqrt{25}$. (3) $\sqrt{26}$. (4) $\sqrt{27}$.

(4) $\sqrt{27}$

$$\vec{v}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\langle e^t, t+3, 5t\rangle = \langle e^t, 1, 5\rangle$$

$$\vec{v}(0) = \langle 1, 1, 5\rangle$$

$$|\vec{v}(0)| = \sqrt{1^2 + 1^2 + 5^2} = \sqrt{27}$$