University of California Davis
Calculus MAT 21C

Name (Print): ______________________
Student ID (Print): ______________________

Practice Midterm Examination
Time Limit: 50 Minutes

May 26 2023

This examination document contains 10 pages, including this cover page, and 4 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (25 points) Consider the points $P = (0, 3, -2)$ and $Q = (4, -3, -11)$.

   (a) (5 points) Find the vector $\vec{PQ}$.

   The components of $\vec{PQ}$ are obtained by taking the coordinates of Q minus the coordinates of P. Therefore $\vec{PQ}$ is

$$\vec{PQ} = \langle 4 - 0, -3 - 3, -11 - (-2) \rangle = \langle 4, -6, -9 \rangle.$$

   (b) (5 points) Show that $|\vec{PQ}| = \sqrt{133}$, i.e. the length equals the square root of 133.

   Since we obtained $\vec{PQ} = \langle 4, -6, -9 \rangle$ in Part (a), the length is

$$|\vec{PQ}| = \sqrt{4^2 + (-6)^2 + (-9)^2} = \sqrt{16 + 36 + 81} = \sqrt{133}.$$

(c) (10 points) Find the midpoint of P and Q.

The midpoint of P and Q is the average of the coordinates from P and Q, or alternatively the endpoint of the vector sum $\vec{OP} + \frac{1}{2} \cdot \vec{PQ}$. This gives the point

$$\text{Midpoint } = \left(\frac{0+4}{2}, \frac{3-3}{2}, \frac{-2-11)}{2}\right) = \left(2, 0, -\frac{13}{2}\right).$$

(d) (5 points) Consider the sphere given by the equation

$$x^2 + (y-3)^2 + (z+2)^2 = 4.$$

For each of the two points P and Q, decide whether they are *inside* the sphere, *on* the sphere or *outside* the sphere.

In general, for a point with coordinate (a, b, c),

$$\text{if } a^2 + (b-3)^2 + (c+2)^2 < 4, \text{ then } (a, b, c) \text{ is inside the sphere;}$$

$$\text{if } a^2 + (b-3)^2 + (c+2)^2 = 4, \text{ then } (a, b, c) \text{ is on the sphere;}$$

$$\text{if } a^2 + (b-3)^2 + (c+2)^2 > 4, \text{ then } (a, b, c) \text{ is outside the sphere.}$$

Now, for $P = (0, 3, -2)$ and $Q = (4, -3, -11)$ we obtain

$$0^2 + (3-3)^2 + (-2+2)^2 = 0 < 4,$$

$$4^2 + (-3-3)^2 + (-11+2)^2 = 16 + 36 + 81 > 4.$$

Therefore, this implies that P is inside the sphere, and Q is outside the sphere. (P is in fact the center of the sphere.)

2. (25 points) Consider the vectors $u = \langle 3, 1, -4\rangle$ and $v = \langle 2, -10, 5\rangle$.

(a) (5 points) Compute the lengths $|u|$ and $|v|$.

Given a vector $\vec{u} = \langle u_1, u_2, u_3\rangle$, the length of $\vec{u}$ equals to $\sqrt{u_1^2 + u_2^2 + u_3^2}$. This gives

$$|u| = \sqrt{3^2 + 1^2 + (-4)^2} = \sqrt{9 + 1 + 16} = \sqrt{26},$$

and

$$|v| = \sqrt{2^2 + (-10)^2 + 5^2} = \sqrt{4 + 100 + 25} = \sqrt{129}.$$

(b) (5 points) Find the dot product $u \cdot v$.

Given $\vec{u} = \langle u_1, u_2, u_3\rangle$, $\vec{v} = \langle v_1, v_2, v_3\rangle$, the dot product

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

Hence,

$$u \cdot v = 3 \cdot 2 + 1 \cdot (-10) + (-4) \cdot 5 = 6 - 10 - 20 = -24.$$

(c) (5 points) Show that the angle θ between u and v satisfies

$$\cos\theta = -4 \cdot \sqrt{\frac{6}{559}}.$$

The dot product formula is $u \cdot v = |u||v|cos\theta$, then

$$cos\theta = \frac{u \cdot v}{|u||v|} = \frac{-24}{\sqrt{26}\sqrt{129}} = -4 \cdot \frac{6}{\sqrt{3354}} = -4 \cdot \sqrt{\frac{36}{3354}} = -4 \cdot \sqrt{\frac{6}{559}}.$$

(d) (5 points) Find the cross product $u \times v$.

Using the cross product formula, in the matrix format, we have

$$\begin{aligned} u \times v &= \begin{vmatrix} i & j & k \\ 3 & 1 & -4 \\ 2 & -10 & 5 \end{vmatrix} = \begin{vmatrix} 1 & -4 \\ -10 & 5 \end{vmatrix} i - \begin{vmatrix} 3 & -4 \\ 2 & 5 \end{vmatrix} j + \begin{vmatrix} 3 & 1 \\ 2 & -10 \end{vmatrix} k \\ &= (5-40)i - (15-(-8))j + (-30-2)k \\ &= \langle -35, -23, -32 \rangle. \end{aligned}$$

(e) (5 points) Find the area of the parallelogram spanned by u and v.

The area of the parallelogram spanned by u and v is the length of $u \times v$, so that

$$\text{Area } = |u \times v| = \sqrt{(-35)^2 + (-23)^2 + (-32)^2} = \sqrt{1225 + 529 + 1024} = \sqrt{2778}.$$

3. (25 points) Consider the point $P = (5, 0, 13)$ and the plane π given by the equation

$$\pi := \{x + 2y - z = 10\}.$$

(a) (5 points) Find a direction vector in the perpendicular direction of π.

Given a plane formula, $Ax+By+Cz = D$, the perpendicular direction of this plane is $\langle A, B, C\rangle$. Since π has formula $x + 2y - z = 10$, one perpendicular direction is $\langle 1, 2, -1\rangle$.

(b) (5 points) Compute the distance from P to π.

If P_0 is any point on the plane with normal vector $\vec{n}$, then the formula of the distance from any point P to the plane is

$$\text{distance} = \left|\overrightarrow{P_0P} \cdot \frac{\vec{n}}{|\vec{n}|}\right|.$$

Pick $P_0 = (10, 0, 0) \in \pi$, then $\overrightarrow{P_0P} = \langle -5, 0, 13\rangle$, and we have the distance from the point P to the plane π is

$$\left|\langle -5, 0, 13\rangle \cdot \frac{\langle 1, 2, -1\rangle}{|\langle 1, 2, -1\rangle|}\right| = \left|\frac{-18}{\sqrt{1^2 + 2^2 + (-1)^2}}\right| = \frac{18}{\sqrt{6}}.$$

(c) (10 points) Find the unique plane π' which contains the point P and the two points $Q = (0, 0, 0)$ and $R = (1, 0, 0)$.

Given $P = (5, 0, 13)$, $Q = (0, 0, 0)$ and $R = (1, 0, 0)$, we have

$$\overrightarrow{QP} = \langle 5, 0, 13\rangle,\ \overrightarrow{QR} = \langle 1, 0, 0\rangle.$$

Then the normal vector is

$$\vec{n} = \overrightarrow{QP} \times \overrightarrow{QR} = \begin{vmatrix} i & j & k \\ 5 & 0 & 13 \\ 1 & 0 & 0 \end{vmatrix} = \langle 0, 13, 0\rangle.$$

Hence, by the plane formula and using the coordinate of point Q, the plane π' has the formula

$$0 \cdot (x - 0) + 13 \cdot (y - 0) + 0 \cdot (z - 0) = 0,$$

which can be simplified to the equation

$$\pi' = \{y = 0\}.$$

(d) (5 points) Find the distance from the point P to the line $L = \pi \cap \pi'$ given by the intersection of π and π'.

Given the normal vector for π is $\vec{n}_1 = \langle 1, 2, -1\rangle$, and the normal vector for π' is $\vec{n}_2 = \langle 0, 1, 0\rangle$, the direction for $L = \pi \cap \pi'$ is

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 0 & 1 & 0 \end{vmatrix} = \langle 1, 0, 1\rangle.$$

The distance formula from P to a line through P_0 parallel to $\vec{v}$ is

$$\text{distance } = \frac{|P_0P \times \vec{v}|}{|\vec{v}|}$$

To find a point on the line L, set $z = 0$ in two plane equations, and this gives the point $P_0 = (10, 0, 0)$. Hence, the distance from P to L is

$$\text{distance } = \frac{|\langle -5, 0, 13\rangle \times \langle 1, 0, 1\rangle|}{|\langle 1, 0, 1\rangle|} = \frac{|\langle 0, 18, 0\rangle|}{\sqrt{2}} = \frac{18}{\sqrt{2}}$$

4. (25 points) For each of the statements below, circle the **unique** correct answer. (You do *not* need to justify your answer.)

   (a) (5 points) The intersection of the sphere $x^2 + y^2 + (z-3)^2 = 1$ with the plane $\pi = \{x - y + z = 3\}$ is:

   (1) Empty. (2) A circle. (3) A disk. (4) A half-space. (5) A line.

   The formula for the sphere indicates it centered at $(0, 0, 3)$, and this point is on π. Hence, the plane cuts through the sphere at its center, so that the intersection is a circle.

   (b) (5 points) A particle moves in space following the trajectory $r(t) = \langle \cos(t), t^2, \sin(t) \rangle$. Its speed at $t = 0$ is:

   (1) 0 (2) 0.5 (3) 1 (4) 1.5 (5) 2.

   Given $r(t) = \langle \cos(t), t^2, \sin(t) \rangle$, the velocity vector function equals

$$v(t) = \langle -\sin(t), 2t, \cos(t) \rangle,$$

   so that $v(0) = \langle 0, 0, 1 \rangle$, and the speed at $t = 0$ is

$$|v(0)| = \sqrt{(0)^2 + (0)^2 + 1^2} = 1$$

(c) (5 points) The distance from $P = (1, 3, -16)$ to the plane $\pi = \{3x+2y+z = -7\}$ is:

(1) 0 (2) 1 (3) $\sqrt{2}$ (4) 2 (5) $\sqrt{3}$.

If P_0 is any point on the plane with normal vector $\vec{n}$, then the formula of the distance from any point P to the plane is

$$\text{distance} = \left| \overrightarrow{P_0P} \cdot \frac{\vec{n}}{|\vec{n}|} \right|.$$

The plane equation gives $\vec{n} = \langle 3, 2, 1 \rangle$, and by setting $x = y = 0$, we get

$$P_0 = (0, 0, -7) \in \pi.$$

Hence, the distance from P to π is

$$\text{distance} = \left| \langle 1, 3, -9 \rangle \cdot \frac{\langle 3, 2, 1 \rangle}{|\langle 3, 2, 1 \rangle|} \right| = \frac{0}{\sqrt{9+4+1}} = 0.$$

Alternatively, just note that $P = (1, 3, -16)$ is contained in the plane $\pi = \{3x + 2y + z = -7\}$, as it satisfies the equation. Therefore the distance is immediately 0 without computations.

(d) (5 points) Given the trajectory $r(t) = \langle 15t+4, -t, \cos(2t) \rangle$, the acceleration vector at time $t = 2\pi$ is given by the vector:

(1) $\langle 15, -1, 0 \rangle$ (2) $\langle 0, 1, 0 \rangle$ (3) $\langle 0, 0, -4 \rangle$ (4) $\langle 0, -1, 2 \rangle$ (5) $\langle 15, -1, 2 \rangle$.

Given $r(t) = \langle 15t + 4, -t, \cos(2t) \rangle$, then

$$v(t) = \langle 15, -1, -2\sin(2t) \rangle,$$

which implies

$$a(t) = \langle 0, 0, -4\cos(2t) \rangle.$$

So that $a(2\pi) = \langle 0, 0, -4 \rangle$.

(e) (5 points) A particle moves with trajectory $r(t) = (t, 6t, 3t)$ along a line. In its entire trajectory (for all t), it intersects the sphere $\{x^2 + y^2 + z^2 = 16\}$:

(1) Never. (2) At 1 point. (3) At 2 points. (4) At 3 points.

Through setting $x = t, y = 6t, z = 3t$, equation

$$x^2 + y^2 + z^2 = 16$$

is equivalent to

$$t^2 + (6t)^2 + (3t)^2 = 16,$$

which gives

$$46t^2 = 16, \text{ so that } t = \pm\sqrt{\frac{16}{46}}.$$

Hence, there are 2 points of intersection.

Alternatively, without any computation, just notice that the line passes through $(0, 0, 0)$ at $t = 0$, which is the center of the sphere. Therefore it must cut the sphere in 2 points.