University of California Davis **Name (Print):** ______________________
Calculus MAT 21C **Student ID (Print):** ______________________

Solved Practice Midterm **April 28 2023**
Time Limit: 50 Minutes

This examination document contains 7 pages, including this cover page, and 4 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

Do not write in the table to the right.

1. (25 points) Consider the sequence (a_n) defined recursively by $a_{n+1} = \sqrt[3]{a_n^2 + a_n + 2}$ and initial condition $a_1 = 0$.

   (a) (15 points) Show that the sequence (a_n) is convergent.

   We will argue that (a_n) is increasing and bounded above. By the Monotone Convergence Theorem, this will prove that (a_n) converges.

   Increasing. For the first two terms we have $a_1 = 0$, $a_2 = \sqrt[3]{2}$ and thus $a_1 < a_2$. In general, we want to argue that $a_n < a_{n+1}$. Note that all the terms a_n are positive and greater than 1 (because $\sqrt[3]{2}$ is greater than 1). The cubic power is increasing, therefore $a_n < a_{n+1}$ if and only if $a_n^3 < a_{n+1}^3$, which is the inequality

   $$a_n^3 < a_n^2 + a_n + 2.$$

   Since the function $x^3 - x^2 - x - 2$ is increasing for $x \geq 1$, the inequality is true for all a_n and thus $a_n < a_{n+1}$.

   Bounded Let us argue that $a_n \leq 2$. This is true for $a_1 = 2$. In general, if we have iteratively argued that $a_n \leq 2$ we need to show that $a_{n+1} \leq 2$ as well. Indeed

   $$a_{n+1} = \sqrt[3]{a_n^2 + a_n + 2} \leq \sqrt[3]{2^2 + 2 + 2} = \sqrt[3]{8} = 2.$$

   (b) (10 points) Find the limit of the sequence (a_n).

   Let $L = \lim_{n\to\infty} a_n$, then $a_{n+1} = \sqrt[3]{a_n^2 + a_n + 2}$ implies that L must satisfy $L = \sqrt[3]{L^2 + L + 2}$. The only real solution of this equation is $L = 2$, therefore $L = 2$.

2. (25 points) Solve the two parts below

   (a) (20 points) For each of the series below, determine whether the series converges or diverges. You *must* justify your answer in detail. If you are applying a certain test, state the name of the test clearly, the steps implementing the test and its outcome. If a sequence converges, you do *not* need to find the limit.

$$1. \sum_{n=1}^{\infty} \frac{\cos(\ln(n))}{12^n}, \qquad 2. \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^5+3n+6}}, \qquad 3. \sum_{n=1}^{\infty} \frac{\ln(n)}{n^n}, \qquad 4. \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}.$$

For (1), the series is absolutely convergent (and so convergent) because

$$\left|\sum_{n=1}^{\infty} \frac{\cos(\ln(n))}{12^n}\right| \leq \sum_{n=1}^{\infty} \frac{1}{12^n}$$

and $\sum_{n=1}^{\infty} \frac{1}{12^n}$ converges because it is geometric with $r = 1/12 < 1$. The direct comparison test implies that $\sum_{n=1}^{\infty} \frac{\cos(\ln(n))}{12^n}$ (absolutely) converges.

For (2), the direct comparison test shows that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^5+3n+6}}$ converges if and only if $\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$. The latter does converge because it is a p-harmonic series with $p = 5/2$ and this is less than 1 (or e.g. via the integral test).

For (3), the root test states that this series converges if the limit $\sqrt[n]{\left|\frac{\ln(n)}{n^n}\right|}$ exists and it is less than 1. Since

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{\ln(n)}{n^n}\right|} = \lim_{n\to\infty} \frac{\sqrt[n]{\ln(n)}}{n} = \lim_{n\to\infty} \frac{1}{n} = 0,$$

since $\lim_{n\to\infty} \sqrt[n]{\ln(n)} = 1$. Since $0 < 1$, the root test proves that the series converges.

For (4), the alternating series test does apply because $\frac{(-1)^n}{\sqrt{n}}$ is decreasing, positive and converging to 0. Therefore the series converges.

(b) (5 points) Discuss for which positive real values of $\alpha \in (0, \infty)$ this series converges:

$$\sum_{n=1}^{\infty} \frac{n^3}{1+n^\alpha}.$$

The direct comparison test shows that this series converges if and only if

$$\sum_{n=1}^{\infty} \frac{n^3}{n^\alpha} = \sum_{n=1}^{\infty} \frac{1}{n^{\alpha-3}}$$

converges. This latter series is a p-harmonic series with $p = \alpha/3$. Thus, the series converges if and only if $\alpha - 3 > 1$. In consequence, the original series converges if and only if $\alpha > 4$. In conclusion, for $\alpha \in (0, 4]$ the series diverges and for $\alpha \in (4, \infty)$ it converges.

3. (25 points) Solve the following parts.

(a) (8 points) Find the Taylor expansion of $x^2\cos(x)$ of order 8 at $x=0$.

The Taylor expansion of $\cos(x)$ at $x=0$ is

$$\cos(x) \overset{x\approx 0}{=} 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}.$$

Therefore, the Taylor expansion of $x^2\cos(x)$ at $x=0$ is

$$x^2\cos(x) \overset{x\approx 0}{=} x^2-\frac{x^4}{2!}+\frac{x^6}{4!}-\frac{x^8}{6!}+\ldots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+2}}{(2n)!}.$$

The expansion at order 8 is thus

$$x^2-\frac{x^4}{2!}+\frac{x^6}{4!}-\frac{x^8}{6!}.$$

(b) (7 points) Find the Taylor expansion of $f(x)=x^2\cos(x^3)$ of order 20 at $x=0$.

By Part (a), the Taylor expansion of $\cos(x^3)$ at $x=0$ is

$$\cos(x^3) \overset{x\approx 0}{=} 1-\frac{x^6}{2!}+\frac{x^{12}}{4!}-\frac{x^{18}}{6!}+\ldots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{6n}}{(2n)!}.$$

Therefore, the Taylor expansion of $x^2\cos(x^3)$ at $x=0$ is

$$x^2\cos(x^3) \overset{x\approx 0}{=} x^2-\frac{x^8}{2!}+\frac{x^{14}}{4!}-\frac{x^{20}}{6!}+\ldots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{6n+2}}{(2n)!}.$$

Hence, the Taylor expansion of $f(x)=x^2\cos(x^3)$ of order 20 at $x=0$ is

$$x^2-\frac{x^8}{2!}+\frac{x^{14}}{4!}-\frac{x^{20}}{6!}.$$

(c) (7 points) Compute the radius of convergence of the Taylor series of $f(x)$ at $x = 0$.

The series $\sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+2}}{(2n)!}$ converges for all $x \in (-\infty, \infty)$. For instance, this is implied by the ratio test applied to this series. In consequence, the radius of convergence is $R = \infty$, i.e. the series converges in the entire real line.

(d) (3 points) What is the approximated value of $f(0.1)$ given by the Taylor approximation of order 20 at $x = 0.1$?

By Part (c) it suffices to evalue

$$x^2 - \frac{x^8}{2!} + \frac{x^{14}}{4!} - \frac{x^{20}}{6!}$$

at $x = 0.1$. Therefore the answer is

$$(0.1)^2 - \frac{(0.1)^8}{2} + \frac{(0.1)^{14}}{24} - \frac{(0.1)^{20}}{720}.$$

(For your information, this value is approximately $0.00999999500000041\ldots$. It is fine to present the answer as the expression above.)

4. (25 points) For each of the ten sentences below, circle whether they are **true** or **false**. (You do *not* need to justify your answer.)

   (a) (5 points) If a sequence (a_n) converges to 0, then $\sum_{n=1}^{\infty} a_n$ converges.

   (1) True. **(2) False.**

   (b) (5 points) If a power series $A(x)$ (centered at $x = 0$) diverges for $x = 12$, then it diverges for $x = -14$.

   (1) True. (2) False.

   (c) (5 points) The Taylor series of a non-constant differentiable function $f : \mathbb{R} \to \mathbb{R}$ at $x = 0$ cannot have all its Taylor coefficients be equal to zero.

   (1) True. **(2) False.**

   (d) (5 points) The Taylor series associated to a real polynomial centered at *any* point $x = a$, $a \in (-\infty, \infty)$, always converges.

   (1) True. (2) False.

   (e) (5 points) If the ratio test can determine the convergence of a convergent series $\sum_{n=1}^{\infty} a_n$, then so can the root test.

   (1) **True.** (2) False.