MAT 1240B: PROBLEM SET 1

DUE TO FRIDAY APR 14 2023

ABSTRACT. This problem set corresponds to the first week of the course MAT-240B Spring 2023. It is due Friday Apr 14 at 9:00pm submitted via Gradescope.

Task: Solve *one* of the problems below and submit it through Gradescope by Friday Apr 14 at 9pm. Be rigorous and precise in writing your solutions.

Problem 1. For each of the following smooth functions $f : \mathbb{R} \longrightarrow \mathbb{R}$, determine whether f is a Morse function or not:

(a) $\cos(x)$.
(b) $x^3 + \varepsilon \cdot x$, depending on the value of $\varepsilon \in \mathbb{R}$.
(c) $\operatorname{tr}\Big((e^{-x^2}, \cos(x)^2, -x^2)^t \cdot (1, 0, -1)\Big)$.
(d) (Optional) $Ai(x) := \frac{1}{\pi}\int_0^\infty \cos(t^3/3 + xt)dt$.

Problem 2. Two Morse functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are said to be *equivalent* is they can be transformed one into the other by smooth changes of coordinates (a diffeomorphism) in the domain and in the target. Find the number $K(n)$ of pairwise non-equivalent functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ having n non-degenerate critical points with pairwise different critical values, supposing that at infinity the function behaves like x for even n and like x^2 for odd n.

(*Answer: The numbers $K(n)$ satisfy $\sum_n K(n)\frac{t^n}{n!} = \sec t + \tan t$.*)

Problem 3. Solve the following parts:

(a) Describe a Morse function in the 2-sphere S^2 with $2n$ critical points, where $n \in \mathbb{N}$ is a positive natural number.
(b) Find a non-constant smooth function $f : S^2 \longrightarrow \mathbb{R}$ which is not a Morse function.
(c) Is it possible for S^2 to have a Morse function with an odd number of critical points?

Problem 4. Let M, N be smooth manifolds and $f : M \longrightarrow \mathbb{R}$ and $g : N \longrightarrow \mathbb{R}$ be Morse functions.

(1) Show that $f + g : M \times N \longrightarrow \mathbb{R}$ is a Morse function.
(2) Compute the number of critical points and indices of $f + g$ in terms of the number of critical points and their indices of f and g.
(3) Construct a Morse function on the n-torus $T^n = S^1 \times \overset{(n)}{\cdots} \times S^1$ with $\binom{n}{k}$ critical points of index k (for a total of 2^n critical points).
(4) (Optional) Find a smooth function $f : T^2 \longrightarrow \mathbb{R}$ with 3 critical points.

Problem 5. Consider the special orthogonal group

$$SO(n) := \{A \in M_{n\times n}(\mathbb{R}) : A \cdot A^t = \mathrm{Id}, \quad \det(A) = 1\}.$$

Let $R \in M_{n\times n}(\mathbb{R})$ be a symmetric matrix with *distinct* positive eigenvalues and consider the function

$$f_R : SO(n) \longrightarrow \mathbb{R}, \quad f_R(A) = -\operatorname{tr}(R \cdot A^t).$$

(i) Show that f_R is a Morse function.
(ii) Compute the number of critical points of f_R and their indices.