MAT 1240B: PROBLEM SET 2

DUE TO FRIDAY MAY 12 2023

Abstract. This problem set corresponds to the third and fourth weeks of the course MAT-240B Spring 2023. It is due Friday May 12 at 9:00pm submitted via Gradescope.

Task: Solve *one* of the problems below and submit it through Gradescope by Friday May 12 at 9pm. Be rigorous and precise in writing your solutions.

For the first problems, consider the 2-sphere $M = S^2$, which can be described as the submanifold $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\} \subseteq \mathbb{R}^3$. Consider the spherical parametrization

$$\sigma(\theta, \phi) = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta), \qquad \theta \in [0, \pi], \varphi \in [0, 2\pi).$$

The real space $(\mathbb{R}^3, g_{flat})$ has the standard Euclidean metric g_{flat} given by the identity at each point, i.e. the standard inner product at each tangent space. It therefore induced a metric g_{S^2} on its unit 2-sphere.

Problem 1. (Round metric and compatible connection) Solve the following parts:

(i) Verify that the metric in S^2 induced as a submanifold of $(\mathbb{R}^3, g_{flat})$ is given in spherical coordinates by

$$g_{S^2} := \begin{pmatrix} 1 & 0 \\ 0 & \cos^2\theta \end{pmatrix}.$$

(ii) Show that the only non-zero Christoffel symbols for the Levi-Civita connection of g_{S^2} are given by

$$\Gamma^1_{22} = \sin\theta\cos\theta, \quad \Gamma^2_{12} = \Gamma^2_{21} = -\tan\theta.$$

(iii) (Optional) Show that there exists *no* metric on S^2 whose associated Levi-Civita connection has all Christoffel symbols equal to 0.

Problem 2. (Geodesics on the round 2-sphere) Let (S^2, g) be the round sphere $g = g_{S^2}$ and ∇ its Levi-Civita connection.

(i) Write the differential equations that a parametrized curve $\gamma : [0, 1] \longrightarrow S^2$ must satisfy to be a geodesic.

(ii) Argue that great circles in S^2 can be parametrized to be geodesics.

Hint: You can do that directly by verifying that they can be parametrized with a parametrization that solve the equations in Part (i). Alternatively, you can try to argue geometrically without any equations using the isometry group $SO(3)$.

(iii) Find all geodesics in (S^2, g_{S^2}).

Problem 3. (Conjugate points in S^2) Let (S^2, g) be the round sphere.

(i) Find all geodesics $\gamma : [0, 1] \longrightarrow S^2$ starting at the North pole $p = (0, 0, 1)$ and ending at the point $q = (1, 0, 0)$.

(ii) For each of the geodesics in Part (i), compute their index as critical points of the energy functional $E : \Omega_{p,q}M \longrightarrow \mathbb{R}$.

Problem 4. (Jacobi fields in S^2) Let (S^2, g) be the round sphere.

(i) Consider the shortest geodesic $\gamma : [0,1] \longrightarrow S^2$ starting at the North pole $p = (0,0,1)$ and ending at South pole $q = (0,0,-1)$. Explicitly write the unique non-zero Jacobi field along γ vanishing at p and q.

(ii) Consider a great circle parametrized as a geodesic $\gamma : [0,1] \longrightarrow S^2$ starting and ending at the North pole $p = (0,0,1)$. Discuss the space of non-zero Jacobi field along γ vanishing at p.

For the next three problems, we consider the flat Euclidean plane $(\mathbb{R}^2, g_{flat})$ and the hyperbolic plane $(\mathbb{H}^2, g_{\mathbb{H}^2})$, where $\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 : y > 0\}$. The metrics are given by

$$g_{flat} = dx \otimes dx + dy \otimes dy, \qquad g_{hyp} = \frac{1}{y^2}\left(dx \otimes dx + dy \otimes dy\right).$$

Problem 5. (Levi-Civita for g_{flat} and g_{hyp}) Solve the following parts:

(i) Show that the Levi-Civita connection associated to g_{flat} has all Christoffel symbols vanishing.

(ii) Show that the only non-zero Christoffel symbols of the Levi-Civita of g_{hyp} are

$$\Gamma^2_{11} = \frac{1}{y}, \quad \Gamma^2_{22} = \Gamma^1_{21} = \Gamma^1_{12} = \frac{-1}{y}.$$

(iii) Show that the geodesic equations for g_{hyp} are

$$x''y = 2x'y', \quad y''y = (y')^2 - (x')^2.$$

Problem 6. (No conjugate points for g_{flat} and g_{hyp}) Solve the following parts:

(i) Show that no two points in $(\mathbb{R}^2, g_{flat})$ and $(\mathbb{H}^2, g_{\mathbb{H}^2})$ can be conjugate, i.e. there are no Jacobi fields (along geodesics) vanishing at both ends.

(In particular, the index of a geodesic is always 0 in these metrics.)

(ii) Suppose that (M, g) is a smooth manifold with all sectional curvatures non-positive. Show that no two points $p, q \in M$ are conjugate.

(In particular, there is no metric in S^2 with all negative sectional curvatures.)

Problem 7. (Parallel transport in S^2 and $\mathbb{H}^2$)

(i) Consider the shortest geodesic $\gamma : [0,1] \longrightarrow S^2$ starting at the North pole $p = (0,0,1)$ and ending at the point $q = (1,0,0)$ and the basis

$$V_1(0) = \partial_x, \quad V_2(0) = \partial_y, V_1(0), V_2(0) \in T_pS^2.$$

of T_pS^2. Find the parallel transport of $V_1(0)$ and $V_2(0)$ along γ.

(ii) Consider the shortest geodesic $\gamma : [0,1] \longrightarrow (\mathbb{H}^2, g_{hyp})$ starting at $p = (0,1)$ and ending at the point $q = (0,3)$ and the basis

$$V_1(0) = \partial_x, \quad V_2(0) = \partial_y, V_1(0), V_2(0) \in T_p\mathbb{H}^2.$$

of $T_p\mathbb{H}^2$. Find the parallel transport of $V_1(0)$ and $V_2(0)$ along γ.