MAT 215B: PROBLEM SET 1

DUE TO FRIDAY APR 12 2024

ABSTRACT. This problem set corresponds to the first week of the course MAT-215B Spring 2024. It is due Friday Apr 12 at 9:00pm submitted via Gradescope.

Task: Solve *two* of the problems below and submit it through Gradescope by Friday Apr 12 at 9pm. Be rigorous and precise in writing your solutions.

Problem 1. First, check that the following chain complexes $(C_\bullet, \partial)$ of $\mathbb{Z}$-modules are indeed chain complexes (i.e. the differential square to zero). Second, compute the homology of each of them.

Notation: A map between $\mathbb{Z}$-modules will be given in its matrix expression, so that the image of an elements is given by left multiplication by that matrix. Elements in a $\mathbb{Z}$-modules are represented by vertical vectors. The 0th graded piece is indicated by [0].

(1)

$$0 \longrightarrow \mathbb{Z} \xrightarrow{(0)} \mathbb{Z}[0] \longrightarrow 0$$

(2)

$$0 \longrightarrow \mathbb{Z} \xrightarrow{(2)} \mathbb{Z}[0] \longrightarrow 0$$

That is, the map $\mathbb{Z} \longrightarrow \mathbb{Z}$ is $x \longmapsto (2) \cdot x = 2x$.

(3)

$$0 \longrightarrow \mathbb{Z} \xrightarrow{(5) \pmod 8} \mathbb{Z}_8[0] \longrightarrow 0$$

That is, the map $\mathbb{Z} \longrightarrow \mathbb{Z}_8$ is reduction modulo 8 and multiplication by 5.

(4)

$$0 \longrightarrow \mathbb{Z} \xrightarrow{(2) \pmod 8} \mathbb{Z}_8[0] \longrightarrow 0$$

(5)

$$0 \longrightarrow \mathbb{Z} \xrightarrow{(0)} \mathbb{Z} \xrightarrow{(7)} \mathbb{Z} \xrightarrow{(1) \pmod 7} \mathbb{Z}_7 \xrightarrow{(0)} \mathbb{Z}_7[0] \longrightarrow 0$$

(6)

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\begin{pmatrix} 3 \\ 1 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} -1 & 3 \\ -2 & 6 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 2 & -1 \end{pmatrix}} \mathbb{Z}[0] \longrightarrow 0$$

Problem 2. *Compute the homology of the following chain complexes of $\mathbb{Z}$-modules:*

(1)

$$0 \longrightarrow \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} 1 & 3 & -1 \\ -1 & 2 & 1 \\ 0 & -1 & 1 \\ 2 & -1 & 0 \end{pmatrix}} \mathbb{Z}^4[0] \longrightarrow 0$$

(2)

$$0 \longrightarrow \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} 5 & 6 & -3 \\ -2 & 4 & 3 \\ 0 & -1 & 0 \end{pmatrix}} \mathbb{Z}^3[0] \longrightarrow 0$$

(3)

$$0 \longrightarrow \mathbb{Z}^4 \xrightarrow{\begin{pmatrix} 4 & 5 & -11 & 2 \\ 1 & -2 & 4 & 4 \end{pmatrix}} \mathbb{Z}^2[0] \longrightarrow 0$$

Problem 3. *Find a Δ-complex structure on each of the following topological spaces X:*

(1) $X = \Sigma_g$, *the orientable genus g surface.*
(2) $X = S^n$, *the n-sphere, $n \in \mathbb{N}$.*
(3) $X = \mathbb{RP}^n$, *the real projective n-space, $n \in \mathbb{N}$.*

Problem 4. *Compute the simplicial homology*[1] *of each of the following topological spaces X:*

(1) $X = S^1$.
(2) $X = \bigvee^n S^1$, *the wedge of n circles.*
(3) $X = \Sigma_g$, *the orientable genus g surface.*
(4) $X = \mathbb{RP}^2$, *the real projective plane.*
(5) $X = \mathbb{RP}^2 \# \mathbb{RP}^2$, *the connected sum of two real projective planes.*

Problem 5. *Find a Δ-complex structure on each of the following topological spaces X:*

(1) *The submanifold* $X = \{(z,w) \in \mathbb{C}^2 : z^2 = w^3\} \cap \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\} \subseteq \mathbb{C}^2$.

(2) *The submanifold* $X = \{[z_1 : z_2 : z_3] \in \mathbb{CP}^2 : z_1 + z_2 - 4z_3 = 0\} \subseteq \mathbb{CP}^2$.

(3) $X = SO(3)$, *the special group of orthogonal* (3×3)*-matrices, i.e.*

$$X = \left\{ A \in M_{3\times 3}(\mathbb{R}) : A \cdot A^t = Id_3, \quad \det(A) = 1 \right\} \subseteq M_{3\times 3}(\mathbb{R}) \cong \mathbb{R}^9.$$

Problem 6. *Compute the simplicial homology of each of the following topological spaces X:*

(1) *The submanifold*
$X = \{(z_1,z_2,z_3) \in \mathbb{C}^3 : z_1^2 + z_2^3 + z^3 = 0\} \cap \{(z_1,z_2,z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\} \subseteq \mathbb{C}^3$.
(2) *The submanifold* $X = \{[z_1 : z_2 : z_3] \in \mathbb{CP}^2 : z_1^2 + z_2^2 + z_3^2 = 0\} \subseteq \mathbb{CP}^2$.

(3) *(Optional) The smooth manifold*

$$Gr_{2,4}(\mathbb{R}) = \{V \subseteq \mathbb{R}^4 : \ V \subseteq \mathbb{R}^4 \text{ oriented vector subspace of dimension } \dim_{\mathbb{R}}(V) = 2\},$$

i.e. the real Grassmannian of oriented 2-planes through the origin inside $\mathbb{R}^4$.

[1] For a choice of Δ-structure, which you get to make.