Lecture 10 : Spans, linear independence & basis (review & practice of chapter II so far)

① Midterm this Friday: 50 min, 2 practice midterm available

Exercise $V = \mathbb{R}^3$, $v_1, v_2, v_3 \in V$ non-zero,

(i) $\text{span}(v_1, v_2, v_3) \overset{?}{=} \text{span}(v_1, v_2, v_1+v_3)$

v_1+v_3 $\quad$ $v_3 = (v_1+v_3) - v_1$ → alternative proof

$v_1 = (1,0,0)$ $v_3 = (0,0,1)$

Sol^n: (i) To get intuition, try: $v_2 = (0,1,0)$

$\text{span}((1,0,0), (0,1,0), (0,0,1)) = V$

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To prove = we would need $\subseteq$ and $\supseteq$

$\boxed{\text{Try } \subseteq}$ $\underbrace{\text{we have } w \in \text{span}(v_1, v_2, v_3)}$, we want $w \in \text{span}(v_1, v_2, v_1+v_3)$ } $\exists a_i'$ s.t. $w = a_1'v_1 + a_2'v_2 + a_3' \cdot (v_3+v_1)$ ✱ (want to prove)

$\exists a_i \in \mathbb{R}$ s.t. $w = a_1v_1 + a_2v_2 + a_3v_3$

given

Can we choose a_i', depending on a_i, s.t. ✱ holds

we have $a_1v_1 + a_2v_2 + a_3v_3 = (a_1' + a_3') \cdot v_1 + a_2'v_2 + a_3'v_3$, $= \binom{b.c}{=w}$

R.M.K.: If $\{v_i\}$ are basis of V, then we'd have $a_1 = (a_1' + a_3')$, $a_2 = a_2'$, $a_3 = a_3'$

we chose: $a_2' = a_2$, $a_3' = a_3$, $a_1' = a_1 - a_3$ → works in general

If $\{v_i\}$ not a basis then, say, v_3 is l.d. with $\{v_1, v_2\}$,

i.e. $v_3 \in \text{span}(v_1, v_2)$ ⟹ L.H.S. = R.H.S. = $\text{span}(v_1, v_2)$

Problem: given subspace $U \subseteq V = \mathbb{R}^3$, find a basis,

basis & span, linear equations

$U_f := \{v \in V : f(v) = 0\}$ where $f: \mathbb{R}^3 \to \mathbb{R}$ $v = (x_1, x_2, x_3) \longmapsto f(v) = (3x_1 - 4x_2 + x_3)$

Sol^n: The map $f: \mathbb{R}^3 \to \mathbb{R}$ is intuitively a projection onto a line,

First, $f(1,0,0) = 3 \neq 0$, ∴ $(1,0,0) \notin U_f$. So $U_f \neq V$

To find a basis for U_f, it suffices to find $v_1, v_2 \in U_f$ s.t. v_1, v_2 are linearly independent

↳ example: choose $x_1 = 0$, $4x_2 = x_3$ } $v_1 = (0,1,4)$

choose $x_2 = 1$, $x_3 = 4$

↳ example: choose $x_2 = 0$ ⟹ $3x_1 + x_3 = 0$ } $v_2 = (1,0,-3)$

choose $x_1 = 1$ $x_3 = -3$

linear independent $v_1 \neq kv_2$

∵ $\dim(U_f) \overset{✱}{\leq} 2$, $\{v_1, v_2\}$ basis

Problem Let $V = \mathbb{R}[x]$, find a basis of $U := \{p(x) \in \mathbb{R}[x] : p(0) = 0,\ p(1) = 0\}$

Sol^n: A basis for V is $\{1, x, x^2, x^3, \ldots, x^{1443}, \ldots\}$ (infinitely many)

U is cut out by 3 eq^n: $p(0) = 0 \Leftrightarrow a_0 = 0$

$p(1) = 0 \Leftrightarrow \sum a_i = 0$

$p(-1) = 0 \Leftrightarrow \sum_{i=0} (-1)^i a_i = 0$

E.g. $p(x) = 1 \notin U$ $\quad x^2 \notin U$

$x \notin U$ $\quad 1-x \notin U$ ... is anybody in U

Try $\quad x - x^3 \in U$

Tweak the basis for V to give a basis for U

Personal writeups:

A basis for V is $\{1, x, x^2, x^3, \ldots, x^{1943}, \ldots\}$ (infinitely many)

U is cut out by 3 eqn : $p(0) = 0 \iff a_0 = 0$

$p(1) = 0 \iff \sum a_i = 0$

$p(-1) = 0 \iff \sum_{i=0} (-1)^i a_i = 0$

i.e. $U = x(x-1)(x+1)(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots)$

$= a_0 (x^3 - x) + a_1 (x^4 - x^2) + a_2 (x^5 - x^3) + \ldots + a_n (x^{3+n} - x^{n+1}) + \ldots$

$\because$ we know that a basis for $V = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots$ is $\{1, x, x^2, \ldots\}$

Replace x^n with $(x^{3+n} - x^{n+1})$ yields,

a basis for $U = a_0 (x^3 - x) + a_1 (x^4 - x^2) + \ldots + a_n (x^{3+n} - x^{n+1}) + \ldots$ is $\{(x^3 - x), (x^4 - x^2), \ldots, (x^{3+n} - x^{n+1}) \ldots\}$