Linear maps

previous knowledge : $f: \mathbb{R}^n \to \mathbb{R}^m$

$$\vec{x} := \underbrace{(x_1, \dots, x_n)}_{n\text{-tuple of } \mathbb{R}} \longmapsto \overset{f(\vec{x})}{\underbrace{f(x_1, x_2, x_3, \dots, x_n)}_{m\text{-tuple of } \mathbb{R}}}$$

(or x)

— known as a map (maps = functions)

Goal of lecture: learn what linear maps are, give examples & non-examples

sums + multiplying by a no.

def: A map $f: \mathbb{R}^n \to \mathbb{R}^m$ is said to be linear if the following 2 conditions are satisfied:

(i) $f(\vec{x}+\vec{y}) = f(\vec{x}) + f(\vec{y})$ ⟵ f commutes with sum

(ii) $f(c \cdot \vec{x}) = c \cdot f(\vec{x})$ ⟵ f commutes with scalar multiplication

any $c \in \mathbb{R}$

Examples: $f: \mathbb{R} \to \mathbb{R}$ could be

$f(x) = 3x$, $f(x) = 4x+2$, $f(x) = x^2 - 2$, $f(x) = e^x$ $\left(f(3+5) = e^{3+5} \neq e^3 + e^5 = f(3) + f(5) \right)$

linear

non-linear

Exercise: Decide wether these maps are linear.

(a) $f: \mathbb{R} \to \mathbb{R}$, $\substack{n=1 \\ m=1}$, $f(x) = 3x$

The 2 conditions are:

For (i), we need to check if $f(x+y) \overset{?}{=} f(x) + f(y)$

L.H.S. $= f(x+y) = 3(x+y)$

R.H.S. $= f(x) + f(y) = 3x + 3y$

L.H.S. = R.H.S. by distribution law

For (ii), we want $f(c \cdot x) = c \cdot f(x)$

L.H.S. $= 3 \cdot (c \cdot x)$

R.H.S. $= c \cdot 3x$

L.H.S. = R.H.S by communtativity of $\mathbb{R}$

(b) $f(x) = 4x - \boxed{2}$, is it linear?

adding a constant is NOT a linear map

For condition (i):

$f(x+y) \overset{?}{=} f(x) + f(y)$

L.H.S. $= 4(x+y) - 2$

R.H.S. $= 4x - 2 + 4y - 2$

$= 4(x+y) - 4$

L.H.S. $\neq$ R.H.S.

=> not satisfy (i), ∴ not linear

Checking (ii):

$f(cx) \overset{?}{=} cf(x)$

L.H.S. $= 4cx - 2$

R.H.S. $= 4cx - 2c$

L.H.S. $\neq$ R.H.S. for all c

(c) $f: \mathbb{R}^2 \to \mathbb{R}^3$,

$f(x_1, x_2) = (3x_1 - 4x_2, x_2 - x_1, 5x_1 x_2)$

Is this linear?

By def, this is asking if **ALL** components $3x_1 - 4x_2, x_2 - x_1, 5x_1 x_2$

are the 3 of them linear

For $3x_1 - 4x_2$: $3(x_1+y_1) - 4(x_2+y_2) = 3x_1 - 4x_2 + 3y_1 - 4y_2$

For $x_2 - x_1$: . . .

However, for $5x_1 x_2$:

L.H.S. $= 5(x_1+y_1)(x_2+y_2)$

R.H.S. $= 5x_1 x_2 + 5y_1 y_2$

L.H.S. $\neq$ R.H.S.

∴ The map does not satisfy (i),

f is not component //

For f to be linear,

all $(f_1, f_2, f_3, \dots f_n)$

must be linear

Lemma (Linear maps compose to linear maps)

hypothesis{ Suppose $f_1 : \mathbb{R}^n \to \mathbb{R}^m$, $f_2 : \mathbb{R}^m \to \mathbb{R}^k$ are linear ← what we have

conclusion{ The composition: $\underbrace{f_2 \circ f_1}_{\text{first apply } f_1, \text{ then } f_2} = f_2(f_1(x)) : \mathbb{R}^n \to \mathbb{R}^k$ is LINEAR

Proof:

What we want is $f_2 \circ f_1$ linear,

need to check the 2 linearity conditions:

(i) $(f_2 \circ f_1)(x+y) \overset{?}{=} (f_2 \circ f_1)(x) + (f_2 \circ f_1)(y)$

$$\begin{aligned} \text{L.H.S.} &= f_2(f_1(x+y)) \\ &= f_2(f_1(x) + f_1(y)) && \boxed{f_1 \text{ linear}} \\ &= f_2 f_1(x) + f_2 f_1(y) && \boxed{f_2 \text{ linear}} \end{aligned}$$

We know by hypothesis,

$$\left.\begin{aligned} f_1(x+y) &= f_1(x) + f_2(y) \\ f_1(cx) &= c f_1(x) \end{aligned}\right\} \text{by def of } f_1 \text{ being a linear map}$$

Similarly,

$$\left.\begin{aligned} f_2(x+y) &= f_2(x) + f_2(y) \\ f_2(cx) &= c f_2(x) \end{aligned}\right\} \text{by def of } f_2 \text{ being a linear map}$$

$\text{R.H.S.} = f_2 f_1(x) + f_2 f_1(y)$

L.H.S. = R.H.S.

(ii) $(f_2 \circ f_1)(c \cdot x) = c(f_2 \circ f_1)(x)$

$$\begin{aligned} \text{L.H.S.} &= f_2(f_1(cx)) \\ &= f_2(c f_1(x)) && \boxed{f_1 \text{ linear}} \\ &= c f_2(f_1(x)) && \boxed{f_2 \text{ linear}} \end{aligned}$$

$\text{R.H.S.} = c f_2(f_1(x))$

∴ L.H.S. = R.H.S.

Lemma is proven