Span & bases

Goal → to understand span, linear independence / dependence

→ to define dimension of vector spaces & basis of vector spaces.

Def: Linear combination

V vector space over field $\mathbb{K}$

given vectors $v_1, \dots, v_m \in V$

we say that a vector $w \in V$ is a linear combination of $v_1, \dots, v_m$ if

$w = a_1 v_1 + \dots + a_m v_m$ for some $a_1, \dots, a_m \in \mathbb{K}$

Examples:

① $u = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $v = \begin{pmatrix}0\\3\\0\end{pmatrix}$

$w = \begin{pmatrix}5\\7\\0\end{pmatrix} = 5u + \frac{7}{3}v$

$\therefore$ w is a linear combination of u & v

② $w_2 = \begin{pmatrix}0\\0\\5\end{pmatrix} \neq a_1 u + a_2 v$ for any $a_1, a_2 \in \mathbb{K}$ $w_2 \notin span(u,v)$

In general, for this example, $\{a_1 u + a_2 v \mid a_1, a_2 \in \mathbb{R}\} = \left\{\begin{pmatrix}x\\y\\0\end{pmatrix} \mid x, y \in \mathbb{K}\right\}$ $(span(u,v))$

Def: Linear span

The linear span of $v_1, \dots, v_m \in V$ is $span(v_1, \dots, v_m) = \left\{a_1 v_1 + \dots + a_m v_m \mid \begin{matrix} a_i \in \mathbb{K} \\ 1 \le i \le m \end{matrix}\right\}$

Example:

$u = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $u_2 = \begin{pmatrix}2\\0\\0\end{pmatrix}$

$span(u) = \left\{a_1 \begin{pmatrix}1\\0\\0\end{pmatrix} \mid a_1 \in \mathbb{K}\right\} = \left\{\begin{pmatrix}a_1\\0\\0\end{pmatrix} \mid a_1 \in \mathbb{K}\right\}$

$span(u, u_2) = \left\{\begin{pmatrix}a_1\\0\\0\end{pmatrix} \mid a_1 \in \mathbb{K}\right\}$

Lemma:

V is a vector space. Then

(1) $v_j \in span(v_1, \dots, v_m)$ $1 \le j \le m$

(2) $span(v_1, \dots, v_m)$ is a subspace of V

(3) If $U \subset V$ is a subspace st $v_1, \dots, v_m \in U$

$\Rightarrow span(v_1, \dots, v_m) \subset U \subset V$

Remark: $span(v_1, \dots, v_m)$ is the smallest subspace containing $v_1, \dots, v_m$

Proof

(1): $0v_1 + \dots + 0v_{j-1} + v_j + 0v_{j+1} + \dots + 0v_m$

$= v_j \Rightarrow v_j \in span(v_1, \dots, v_m)$

(2) Note that

$0v_1 + \dots + 0v_m = 0$

So $\boxed{0 \in span(v_1, \dots, v_m)}$

$(\exists\, 0)$

$w_1 = a_1 v_1 + \dots + a_m v_m$

$w_2 = b_1 v_1 + \dots + b_m v_m$

$\boxed{w_1 + w_2 = (a_1 + b_1) v_1 + \dots + (a_m + b_m) v_m \in span(v_1, \dots, v_m)}$ $(u_1 + u_2 \in U)$

Also,

(for any $\alpha \in \mathbb{K}$, $\alpha u \in U$)

$\boxed{\alpha w_1 = \alpha a_1 v_1 + \alpha a_2 v_2 + \dots + \alpha a_m v_m \in span(v_1, \dots, v_m)}$

$\Rightarrow span(v_1, \dots, v_m) \subset V$

Def: dimension of a vector space

If $V = \operatorname{span}(v_1, \dots, v_m)$

$\Rightarrow$ we say that $v_1, \dots, v_m$ span V and that V is finite dimensional $(\dim V \le m)$

otherwise (if $V \neq \operatorname{span}(v_1, \dots, v_m)$ for any $v_1, \dots, v_m \in V$) (for any $m \in \mathbb{N}$)

$\Rightarrow$ we say V is infinite dimensional

Example: $p(z) = a_m z^m + \dots + a_1 z + a_0 \quad a_m \neq 0$

$F^m(z) = \{ p(z) \in \mathbb{K}[z] \mid \deg(p(z)) \le m \}$ $\quad \left(\mathbb{K}(z) = \left\{ p(z) \,\middle|\, \begin{array}{l} p(z) = a_m z^m + \dots + a_1 z^1 + a_0 \\ \text{for some } m \in \mathbb{N},\ a_i \in \mathbb{K} \end{array} \right\} \right)$

$\operatorname{span}(\quad) \overset{?}{=} F^m(z)$

$F^0(z) = \{ a_0 \mid a_0 \in \mathbb{K} \} = \operatorname{span}(1)$

$F^1(z) = \{ a_1 z + a_0 \mid a \in \mathbb{K} \} = \operatorname{span}(z, 1)$

$F^2(z) = \{ a_2 z^2 + a_1 z + a_0 \mid a_i \in \mathbb{K} \} = \operatorname{span}(z^2, z, 1)$

$F^m(z) = \operatorname{span}(z^m, z^{m-1}, \dots, z, 1)$ (finite dimensional)

$F^0(z) \subsetneq F^1(z) \subsetneq F^2(z) \subsetneq \dots \subsetneq F^m(z)$

RMK $\mathbb{K}(z)$ is infinite dimensional,

$F^m(z) \simeq \mathbb{K}^{m+1}$

$p(z) = 3z^2 + 4z + 1 \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Question: What is the smallest list of vectors $v_1, \dots, v_m$ such that $U = \operatorname{span}(v_1, \dots, v_m)$?

$\rightarrow$ Recall $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \neq a_1 \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \notin \operatorname{span}\left(\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} \right)$

Def: Linearly independent

A vector $v \in V$ is linearly independent of a set of vectors $v_1, \dots, v_m$ if

$v \neq a_1 v_1 + \dots + a_m v_m$ for any $a_i \in \mathbb{K}$

$v \notin \operatorname{span}(v_1, \dots, v_m)$

we say that v is linearly dependent of a set of vectors $v_1, \dots, v_m$ if $\exists\, a_i \in \mathbb{K}$ s.t.

$v = a_1 v_1 + \dots + a_m v_m$ i.e. $v \in \operatorname{span}(v_1, \dots, v_m)$

Example:

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ dependent $\qquad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \notin \operatorname{span}\left(\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} \right)$ independent

Def: Linear independency of sets of vectors

We say a list of vectors $v_1, \dots, v_m$ is linearly independent if the only solution for $a_1, \dots, a_m \in \mathbb{K}$ of $a_1 v_1 + \dots + a_m v_m = 0$ is $a_i = 0$

Remark: if $v_1 = b_2 v_2 + \dots + b_m v_m$ (if v_1 is linearly dependent)

$v_1 - b_2 v_2 - \dots - b_m v_m = 0$

does not hold true

Example

$a_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + a_1 \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 3a_1 \\ 5a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ independent

as $a_1, a_2, a_3 = 0$

$a_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + a_3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\Rightarrow$ not linearly independent

a_1, a_2, a_3 can be $\begin{cases} a_1 = 1 \\ b_1 = 1 \\ a_3 = -1 \end{cases}$

Def: linearly dependency of a set of vectors

we say $v_1, \dots, v_m$ are a list of linearly dependent vectors if there exists $a_1, \dots a_m$ st $a_i \neq 0$
& $a_1 v_1 + \dots + a_m v_m = 0$

(if some vectors in a set are linearly dependent, the whole set is linearly dependent, ie. $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ are linearly dependent)