Recall:

$\nearrow v_1, \dots, v_m \in V$

tool to construct subspaces $\rightarrow$ $\text{span}(v_1, \dots, v_m) = \{a_1 v_1 + \dots + a_m v_m \mid a_i \in \mathbb{F}\}$

$\hookleftarrow$ smallest subspace containing $v_1, \dots, v_m$

Def^n: $v_1, \dots, v_m$ was a linearly independent list of vectors if the only solution $a_1 v_1 + \dots + a_m v_m = 0$ is $a_i = 0$

If $\exists$ a solution where same $a_i \neq 0$

$\rightarrow$ we say $v_1, \dots, v_m$ is linearly dependent

$$\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}3\\0\end{pmatrix}$$

$$\begin{pmatrix}1\\0\end{pmatrix} - \frac{1}{3}\begin{pmatrix}3\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\text{span}\left(\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right)$$

$$\begin{pmatrix}3\\0\end{pmatrix} = a_1\begin{pmatrix}1\\-1\end{pmatrix} + a_2\begin{pmatrix}0\\1\end{pmatrix} \quad a_1 = 3, \ a_2 \text{ must also} = 3$$

$$\boxed{\begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}} \longrightarrow a_1\begin{pmatrix}1\\-1\end{pmatrix} + a_2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}a_1\\-a_1+a_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

- To check linear independence $\rightarrow$ solve a homogeneous system of linear equations

Why do we care?

LEMMA 5.2.6 The list of vectors $v_1, \dots, v_m$ is linearly independent if and only if for every $v \in \text{span}(v_1, \dots, v_m)$ v can be written uniquely as a linear combination of $v_1, \dots, v_m$

$$\begin{pmatrix}3\\0\end{pmatrix} \in \text{span}\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}3\\0\end{pmatrix}\right)$$

$$\begin{pmatrix}3\\0\end{pmatrix} = a_1\begin{pmatrix}1\\0\end{pmatrix} + a_2\begin{pmatrix}3\\0\end{pmatrix} \rightarrow a_1 = 0, \ a_2 = 1$$

Proof:

($\Rightarrow$) Suppose $v_1, \dots, v_m$ linearly independent and that we can write $v \in \text{span}(v_1, \dots, v_m)$ in two different ways

$v = a_1 v_1 + \dots + a_m v_m \quad a_i \in \mathbb{F}$

$v = b_1 v_1 + \dots + b_m v_m \quad b_i \in \mathbb{F}$

$$0 = \underbrace{(a_1 - b_1)}_{c_1} v_1 + \dots + \underbrace{(a_m - b_m)}_{c_m} v_m$$

Since $v_1, \dots, v_m$ linearly independent,

$a_i - b_i = 0$ for all $1 \leq i \leq m$

$\Rightarrow a_i = b_i \quad 1 \leq i \leq m$

($\Leftarrow$) Suppose for any $v \in \text{span}(v_1, \dots, v_m)$ we can write v uniquely as a linear combination of $v_1, \dots, v_m$

since $0 \in \text{span}(v_1, \dots, v_m)$ and $0 = 0v_1 + \dots + 0v_m$

$\rightarrow$ $0 = a_1 v_1 + \dots a_m v_m$ by my hypothesis $a_i = 0$

Q_1: How can I tell if a list of vectors is dependent?

Q_2: Can I turn a list of dependent vectors into a linearly independent list of vectors

Lemma: If $v_1, \dots, v_m$ linearly dependent and $v_1 \neq 0$

$\Rightarrow$ $\exists$ an index $j \in \{2, \dots, m\}$ s.t. the following prop. hold

1. $v_j \in \text{span}(v_1, \dots, v_{j-1})$
2. $\text{span}(v_1, \dots, \hat{v}_j, \dots, v_m) = \text{span}(v_1, \dots, v_m)$

   $\text{span}(v_1, v_3) = \text{span}(v_1, v_2, v_3)$

Ex.) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + (-1/2)\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \in \text{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right)$$

Remove $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

$$\text{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right) = \text{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}\right)$$

Proof:

$v_1, \dots, v_m$ dependent

$\Rightarrow$ $\exists\, a_i \in \mathbb{F}$ s.t. $a_i \neq 0$ for all $1 \leq i \leq m$

$0 = a_1 v_1 + \dots + a_m v_m$

$v_1 \neq 0$ so $a_2, \dots, a_m \neq 0$ — largest in terms of index where $a_i \neq 0$ not value

Let j be the largest index between 2 and m such that $a_i \neq 0$

$0 = a_1 v_1 + a_2 v_2 + \dots + a_j v_j + 0 v_{j+1} + \dots + 0 v_m$

$\Rightarrow$ $a_j v_j = -a_1 v_1, \dots, -a_{j-1} v_{j-1}$

$v_j = \frac{-a_1}{a_j} v_1, \dots, \frac{-a_{j-1}}{a_j} v_{j-1}$

$\Rightarrow$ $v_j \in \text{span}(v_1, \dots, v_{j-1})$

$v \in \text{span}(v_1, \dots, v_m)$

$v = a_1 v_1 + \dots + a_j v_j + \dots + a_m v_m$

since $v_j \in \text{span}(v_1, \dots, v_{j-1})$, $v_j = c_1 v_1 + \dots + c_{j-1} v_{j-1}$

$\Rightarrow$ $v = a_1 v_1 + \dots + a_{j-1} v_j + j c_1 v_1 + \dots + c_{j-1} v_{j-1} + a_{j+1} v_{j+1} + \dots + a_m v_m = (a_1 + j c_1) v_1 + \dots + (a_{j-1} + c_{j-1}) v_{j-1} + a_{j+1} v_{j+1} + \dots + a_m v_m$

$\Rightarrow$ $v \in \text{span}(v_1, \dots, \hat{v}_j, \dots, v_m)$

Theorem: Let V be a finite dimensional vector space.

Suppose $v_1, \dots, v_m$ is a linearly independent list of vectors that spans V. Let $(w_1, \dots, w_n)$ be a linearly dependent list of vectors that spans V.

$n > m$ $\qquad$ $m = \dim V$

$$\operatorname{span}\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)=\mathbb{R}^2$$

$$\begin{pmatrix}x\\y\end{pmatrix}=(x-y)\begin{pmatrix}1\\0\end{pmatrix}+y\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}x-y+y\\y\end{pmatrix}$$

$$\operatorname{span}\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix}\right)=\begin{pmatrix}2\\1\end{pmatrix}=(2-1)\begin{pmatrix}1\\0\end{pmatrix}+1\begin{pmatrix}1\\1\end{pmatrix}$$