Theorem:

Let $V$ be a finite dimension v.s. if $v_1, \dots, v_n$ is a linear independent spanning list of vector for $V$ & $w_1, \dots, w_m$ is a linearly dependent spanning list of vectors for $V \Rightarrow m > n$

Example:

$\text{span}\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right) = \mathbb{F}^2 = \text{span}\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}\right)$

Proof:

$S_0 = (w_1, \dots, w_m)$ $V = \text{Span}(S_0)$

Step 1 $S_0$ add $v_1$

$\text{span}(\underbrace{v_1, w_1, \dots, w_m}_{\text{linearly dependent}}) = V$

$\exists\, w_{i_1}$ $1 \le i_1 \le m$ st. $w_{i_1} = a_1 v_1 + \dots + \hat{a}_{i_1} \hat{w}_{i_1} + \dots + a_m w_m$

$S_1 = (v_1, w_1, \dots, \hat{w}_{i_1}, \dots w_m)$

Step k Add $v_k$ to $S_{k-1}$

$\text{span}(\underbrace{v_1, \dots, v_k, w_1, \dots, w_{i_j}, \dots, \hat{w}_{i_{k-1}}, \dots, w_m}_{\text{Linear dependent}}) = V$

So I can throw out a $w_{i_k}$ & still have a spanning list of vectors.

Step k if $k-1 = m$

$S_{k-1} = (v_1, \dots, v_{k-1}, \hat{w}_{i_1}, \dots, \hat{w}_m)$

$\Rightarrow \text{span}(v_1, \dots, v_{k-1}) = V$ $\because v_k \in \text{span}(v_1, \dots, v_{k-1})$

$\Rightarrow v_1, \dots, v_k$ linear dependent ($v_1, \dots, v_k$ linearly independent)

(contradiction)

Example.

$\text{span}\left(\underset{v_1}{\begin{pmatrix}1\\0\end{pmatrix}}, \underset{v_2}{\begin{pmatrix}1\\1\end{pmatrix}}\right) = \mathbb{F}^2 = \text{span}\left(\underset{w_1}{\begin{pmatrix}1\\0\end{pmatrix}}, \underset{w_2}{\begin{pmatrix}1\\1\end{pmatrix}}, \underset{w_3}{\begin{pmatrix}2\\1\end{pmatrix}}\right)$

$S_0 = w_1, w_2, w_3$

$S_1 = v_1, \cancel{w_1}, w_2, w_3$

$S_2 = v_2, v_1, \cancel{w_2}, w_3$

$\Rightarrow 2 < 3$

$n < m$

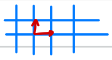
<u>Def: Bases</u>

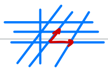
$\boxed{V \text{ finite dimension vector space. A } \underline{\text{linearly independent}}\ \underline{\text{spanning}} \text{ list of vectors for } V \text{ is a basis of } V}$

Examples

$\mathbb{F}^n$ $\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\\vdots\end{pmatrix}, \ldots, \begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}$

$\quad e_1 \qquad e_2 \qquad\qquad e_n$

For $\mathbb{F}^2$ $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} = B_1 \rightarrow$ 

$\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} = B_2 \rightarrow$ 

$\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\} = B_3$

Example:

$\mathbb{F}^m[z] = \{p(z) \in \mathbb{F}[z] \mid \deg(p(z)) \leq m\}$

Basis: $(1, z, z^2, \ldots, z^m)$

Recall if $V = \operatorname{span}(v_1, \ldots, v_n)$

$\boxed{\forall v \in V,\ v \text{ linear combination of } v_1, \ldots, v_n}$

if $v_1, \ldots, v_n$ linear independent

$v = a_1 v_1 + \ldots + a_n v_n$ is $\boxed{\text{unique}}$ $\nearrow$ we define $v_1, \ldots, v_n$ as lin. independent if $\exists$ unique way to decompose $\vec{0}$

Example:

$\vec{v}_1, \vec{v}_2$ basis of $V$ (assumption: $\forall \vec{v} \in V$, $\vec{v} = a\vec{v}_1 + b\vec{v}_2$; $\vec{v}_1, \vec{v}_2$ lin. independent)

Q: $\vec{v}_1 + \vec{v}_2, \vec{v}_2$ a basis of $V$?

$\vec{v}_1 = (\vec{v}_1 + \vec{v}_2) - \vec{v}_2$

$\therefore \operatorname{span}(\vec{v}_1 + \vec{v}_2, \vec{v}_2) = \mathbb{R}^2$

$0 = b_1(\vec{v}_1 + \vec{v}_2) + b_2\vec{v}_2$ $\quad \because 0 = a_1\vec{v}_1 + a_2\vec{v}_2$ has only sol $a_1 = 0, a_2 = 0$

$= b_1\vec{v}_1 + (b_1 + b_2)\vec{v}_2$ $\quad \therefore b_1 = a_1 = 0,$

$b_1 + b_2 = a_2 = 0 \Rightarrow b_2 = 0$

$\therefore \vec{v}_1 + \vec{v}_2, \vec{v}_2$ is linearly independent

<u>Theorem: Basis reduction theorem</u>

$\boxed{\text{If } V = \operatorname{span}(v_1, \ldots, v_m) \Rightarrow v_1, \ldots, v_m \text{ give a basis of } V \text{ after removing some } v_i}$

Example:

$\begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}2\\-2\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$

$\quad v_1 \qquad v_2 \qquad v_3 \qquad v_4 \qquad v_5$

<u>Stepwise</u>

① $v_2 \in \operatorname{span}(v_1)$ $\therefore$ Remove $v_2$,

② $v_3 \notin \operatorname{span}(v_1)$ $\therefore$ keep $v_3$

③ $v_4 \in \operatorname{span}(v_1, v_3)$ $\therefore$ Remove $v_4$

④ $v_5 \notin \operatorname{span}(v_1, v_3)$ $\therefore$ Keep $v_5$

Thus, basis: $v_1, v_3, v_5$

(goal: chop $(v_1, \ldots, v_n)$ spanning list into a linear spanning list $\operatorname{span} = \mathbb{F}^3$)

## Theorem: Basis extension theorem

Every linearly independent list of vectors $v_1, \ldots, v_k$ for $V$ finite dimension vector space can be extended to a basis

Proof:

$V = \text{span}(w_1, \ldots, w_n) \leftarrow$ Linear independent $w_1, \ldots, w_n$

throw in $v_j$:

$\text{span}(v_1, w_1, \ldots \hat{w}_{ij}, \ldots, w_n) = V$

$\text{span}(v_1, \ldots, v_k, w_1, \ldots, w_n, \hat{w}_{ij}, \ldots, \hat{w}_{ik}) = V$

Example:

$\underbrace{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}}_{v_1}, v_2, v_3$

$\mathbb{F}^3 : \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{w_1}, \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{w_2}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{w_3}$

( consider $v_1, w_1, w_2, w_3$:

$w_1 \notin \text{span}(v_1)$, keep $w_1$,

$w_2 \in \text{span}(v_1, w_1)$, remove $w_2$,

$w_3 \notin \text{span}(v_1, w_1)$, keep $w_3$ )

basis: $v_1, w_1, w_3$

## Theorem

If $V$ is finite dimensional $\Rightarrow$ any bases of $V$ have the same length

Proof:

$v_1, \ldots, v_m$ span $V$ linearly independent

$w_1, \ldots, w_n$ span $V$ linearly independent

from basis extension theorem since $v_1, \ldots, v_m$ linear independent

$m \le n$

& by the same theorem $\because w_1, \ldots, w_n$ linear independent:

$n \le m \quad \therefore \boxed{m = n}$

Def:

Length of a basis of a vs $V$ finite dimension is its dimension

Example:

$\mathbb{F}^n$, $e_1, \ldots, e_n$

$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \Rightarrow \dim \mathbb{F}^n = n$

## Theorem:

Spanning set of length $n$

$n = \dim V$

$\Rightarrow v_1, \ldots, v_n$ linear independent

If $v_1, \ldots, v_n$ linear independent,

$\Rightarrow v_1, \ldots, v_n$ must span $V$