MAT 67: PROBLEM SET 1

DUE TO FRIDAY APR 12 2024

Abstract. This problem set corresponds to the first week of the course MAT-67 Spring 2024. It is due Friday Apr 12 at 9:00am submitted via Gradescope.

Purpose: The goal of this assignment is to acquire the necessary skills to work with linear maps. These were discussed during the first week of the course and are covered in Chapter 1 of the textbook.

Task: Solve Problems 1 through 4 below.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

You are welcome to use the Office Hours offered by the Professor and the TA. Again, list any collaborators or contributors in your solutions. Make sure you are using your own thought process and words, even if an idea or solution came from elsewhere. (In particular, it might be wrong, so please make sure to think about it yourself.)

Grade: Each graded Problem is worth 25 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct. If you are using theorems in lecture and in the textbook, make that reference clear. (E.g. specify name/number of the theorem and section of the book.)

Problem 1. For each of the following maps, decide if the map is *linear* or *non-linear* and **prove it**. Each item is worth 5 points:

(1) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x_1, x_2) = (x_1 + x_2, 3x_2)$,
(2) $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$, $f(x_1, x_2, x_3) = (3x_1 - x_2 + x_3, x_1 - x_2 + 4x_3, x_1 - 2x_3 + 1, x_1 - x_2)$,
(3) $f : \mathbb{R} \longrightarrow \mathbb{R}^3$, $f(x) = (4x, |x|, 2)$,
(4) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $f(x_1, x_2) = (x_1 - 7x_2, 3x_1 + x_2, 4x_1 - 9x_2)$.
(5) $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$, $f(x_1, x_2, x_3) = x_1 + x_2 + x_3$

Problem 2. Consider the map $R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$R_\theta(x_1, x_2) := (\cos\theta \cdot x_1 + \sin\theta \cdot x_2, -\sin\theta \cdot x_1 + \cos\theta \cdot x_2),$$

where $\theta \in \mathbb{R}$ is a fixed angle. Each item is worth 5 points. Solve the following parts:

(1) Show that the map R_θ is linear.

(2) What are the values of $R_\theta(1, 0)$,$R_\theta(0, 1)$ and $R_\theta(1, 1)$?

(3) Show that the map is injective.

(4) Show that the map is surjective.

(5) In words, and possibly using a picture, describe what the map R_θ is doing *geometrically* when applied to points (x_1, x_2) in the plane $\mathbb{R}^2$.

(6) (*Optional, extra 5 points*) Show that the composition $R_\theta \circ R_{-\theta}$ is the identity map, i.e. $(R_\theta \circ R_{-\theta})(x_1, x_2) = (x_1, x_2)$. (This is also true for $R_{-\theta} \circ R_\theta$.)

Problem 3. Consider the two maps

$$\pi_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2, \quad \pi_1(x_1, x_2, x_3) := (x_1, x_2 + 4x_3), \text{ and}$$
$$\pi_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3, \quad \pi_2(q_1, q_2) := (q_1 + 2q_2, q_2, q_1 - q_2).$$

Consider the composition $f = \pi_2 \circ \pi_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$. Each item is worth 5 points. Solve the following parts:

(1) Show that f is linear.

(2) What are the values of $f(1, 0, 0)$,$f(0, 1, 0)$ and $f(0, 0, 1)$?

(3) Is there any non-zero point $(a, b, c) \in \mathbb{R}^3$ such that $f(a, b, c) = (0, 0, 0)$?

(4) Show that the map f is *not* injective.

(5) Show that the map f is *not* surjective.

Problem 4. Solve each of the following parts. Each item is worth 5 points:

(1) Find a linear map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $f(1,0) = (4,5)$ and $f(0,1) = (3,-2)$.

(2) Suppose that $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear map such that $f(1,0) = (4,5)$ and $f(0,1) = (3,-2)$. Find $f(5,7)$.

(3) Find two distinct linear maps $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $f(1,0) = (4,5)$ and $g(1,0) = (4,5)$.

(4) Show that there is no linear map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $f(1,1) = (2,-3)$ and $f(0,1) = (5,6)$ and $f(2,1) = (10,4)$.

(5) Is there a linear map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $f(1,0) = (0,0)$ and f is surjective?