MAT 67: PROBLEM SET 1

DUE TO FRIDAY APR 12 2024

Abstract. This problem set corresponds to the first week of the course MAT-67 Spring 2024. It is due Friday Apr 12 at 9:00am submitted via Gradescope. Solutions were typed by TA Scroggin, please contact *tmscroggin – at – ucdavis.edu* for any comments.

Purpose: The goal of this assignment is to acquire the necessary skills to work with linear maps. These were discussed during the first week of the course and are covered in Chapter 1 of the textbook.

Task: Solve Problems 1 through 4 below.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

You are welcome to use the Office Hours offered by the Professor and the TA. Again, list any collaborators or contributors in your solutions. Make sure you are using your own thought process and words, even if an idea or solution came from elsewhere. (In particular, it might be wrong, so please make sure to think about it yourself.)

Grade: Each graded Problem is worth 25 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct. If you are using theorems in lecture and in the textbook, make that reference clear. (E.g. specify name/number of the theorem and section of the book.)

Problem 1. For each of the following maps, decide if the map is *linear* or *non-linear* and **prove it**. Each item is worth 5 points:

(1) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x_1, x_2) = (x_1 + x_2, 3x_2)$,
(2) $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$, $f(x_1, x_2, x_3) = (3x_1 - x_2 + x_3, x_1 - x_2 + 4x_3, x_1 - 2x_3 + 1, x_1 - x_2)$,
(3) $f : \mathbb{R} \longrightarrow \mathbb{R}^3$, $f(x) = (4x, |x|, 2)$,
(4) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $f(x_1, x_2) = (x_1 - 7x_2, 3x_1 + x_2, 4x_1 - 9x_2)$.
(5) $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$, $f(x_1, x_2, x_3) = x_1 + x_2 + x_3$

Solution.

(1) This map is $\boxed{linear}$.

We show that the function satisfies both vector addition and scalar multiplication. First, we check vector addition, i.e., we want to show that $f(x_1 + y_1, x_2 + y_2) = f(x_1, x_2) + f(y_1, y_2)$.

$$\begin{aligned} f(x_1 + y_1, x_2 + y_2) &= ((x_1 + y_1) + (x_2 + y_2), 3(x_2 + y_2)) \\ &= (x_1 + x_2 + y_1 + y_2, 3x_2 + 3y_2) \\ &= (x_1 + x_2, 3x_2) + (y_1 + y_2, 3y_2) \\ &= f(x_1, x_2) + f(y_1, y_2). \end{aligned}$$

Now, we want to show that the map respects scalar multiplication, i.e., $f(c \cdot (x_1, x_2)) = c \cdot f(x_1, x_2)$.

$$\begin{aligned} f(cx_1, cx_2) &= (cx_1 + cx_2, 3cx_2) = (c(x_1 + x_2), c(3x_2)) \\ &= c \cdot (x_1 + x_2, 3x_2) \\ &= c \cdot f(x_1, x_2). \end{aligned}$$

Since the map satisfies both of these conditions then the map is linear.

(2) This map is $\boxed{non-linear}$.

We show that the map fails to satisfy vector addition and scalar multiplications. We start with vector addition.

$$\begin{aligned} f(x_1 + y_1, x_2 + y_2, x_3 + y_3) &= (3(x_1 + y_1) - (x_2 + y_2) + (x_3 + y_3), \\ &\qquad (x_1 + y_1) - (x_2 + y_2) + 4(x_3 + y_3), \\ &\qquad (x_1 + y_1) - 2(x_3 + y_3) + 1, (x_1 + y_1) - (x_2 + y_2)) \\ &= (3x_1 - x_2 + x_3 + 3y_1 - y_2 + y_3, x_1 - x_2 + 4x_3 + y_1 \\ &\qquad - y_2 + 4y_3, x_1 - 2x_3 + y_1 - 2y_3 + 1, x_1 - x_2 + y_1 - y_2) \\ f(x_1, x_2, x_3) + f(y_1, y_2, y_3) &= (3x_1 - x_2 + x_3, x_1 - x_2 + 4x_3, x_1 - 2x_3 + 1, x_1 - x_2) \\ &\quad + (3y_1 - y_2 + y_3, y_1 - y_2 + 4y_3, y_1 - 2y_3 + 1, y_1 - y_2) \\ &= (3x_1 - x_2 + x_3 + 3y_1 - y_2 + y_3, x_1 - x_2 + 4x_3 + y_1 \\ &\qquad - y_2 + 4y_3, x_1 - 2x_3 + y_1 - 2y_3 + 2, x_1 - x_2 + y_1 - y_2). \end{aligned}$$

Here, we see that $f(x_1 + y_1, x_2 + y_2, x_3 + y_3) \neq f(x_1, x_2, x_3) + f(y_1, y_2, y_3)$ since in the third coordinate for the constant terms $1 \neq 2$. Now, we show that scalar multiplication is also not respected.

$$\begin{aligned}
f(cx_1, cx_2, cx_3) &= (3cx_1 - cx_2 + cx_3, cx_1 - cx_2 + 4cx_3, cx_1 - 2cx_3 + 1, cx_1 - cx_2) \\
c \cdot f(x_1, x_2, x_3) &= c \cdot (3x_1 - x_2 + x_3, x_1 - x_2 + 4x_3, x_1 - 2x_3 + 1, x_1 - x_2) \\
&= (3cx_1 - cx_2 + cx_3, cx_1 - cx_2 + 4cx_3, cx_1 - 2cx_3 + c, cx_1 - cx_2).
\end{aligned}$$

Again, the constant in the third coordinate forces $f(c \cdot (x_1, x_2, x_3)) \neq c \cdot f(x_1, x_2, x_3)$. Therefore, the map is non-linear.

(3) This map is $\boxed{non - linear}$.

Similar to previous problem, we may have used the definition of linearity to disprove vector addition and scalar multiplicity. However, we may also show that this map is non-linear by counterexample. More specifically, we show that this map does not satisfy the scalar multiplication condition.

Let $x = 5$ and $c = -1$, then

$$f(-1 \cdot 5) = f(-5) = (4(5), |-5|, 2) = (20, 5, 2),$$

whereas,

$$-1 \cdot f(5) = -1 \cdot (4(5), |5|, 2) = -1 \cdot (20, 5, 2) = (-20, -5, -2).$$

Clearly, $(20, 5, 2) \neq (-20, -5, -2)$, therefore, scalar multiplication is not respected under the map f.

Alternatively, we may have found a counter example to vector addition. For instance we have that $1 + 1 = 2$ and

$$f(2) = (4(2), |2|, 2) = (8, 2, 2),$$

$$f(1) + f(1) = (4(1), |1|, 2) + (4(1), |1|, 2) = (4, 1, 2) + (4, 1, 2) = (8, 2, 4).$$

(4) This map is $\boxed{linear}$.

We show this by confirming that vector addition and scalar multiplication are respected. We start with vector addition:

$$\begin{aligned}
f(x_1 + y_1, x_2 + y_2) &= ((x_1 + y_1) - 7(x_2 - y_2), 3(x_1 + y_1) + (x_2 + y_2), 4(x_1 + y_1) - 9(x_2 + y_2)) \\
&= (x_1 + y_1 - 7x_2 - 7y_2, 3x_1 + 3y_1 + x_2 + y_2, 4x_1 + 4y_1 - 9x_2 - 9y_2) \\
&= (x_1 - 7x_2, 3x_1 + x_2, 4x_1 - 9x_2) + (y_1 - 7y_2, 3y_1 + y_2, 4y_1 - 9y_2) \\
&= f(x_1, x_2) + f(y_1, y_2).
\end{aligned}$$

Now, scalar multiplication:

$$\begin{aligned}
f(c \cdot (x_1, x_2)) &= f(cx_1, cx_2) \\
&= (cx_1 - 7cx_2, 3cx_1 + cx_2, 4cx_1 - 9cx_2) \\
&= (c(x_1 - 7x_2), c(3x_1 + x_2), c(4x_1 - 9x_2)) \\
&= c \cdot (x_1 - 7x_2, 3x_1 + x_2, 4x_1 - 9x_2).
\end{aligned}$$

(5) This map is $\boxed{linear}$.

We confirm this b checking vector addition and scalar multiplication as in the previous problems.

$$\begin{aligned} f(x_1+y_1, x_2+y_2, x_3+y_3) &= (x_1+y_1)+(x_2+y_2)+(x_3+y_3) \\ &= x_1+x_2+x_3+y_1+y_2+y_3 \\ &= (x_1+x_2+x_3)+(y_1+y_2+y_3) \\ &= f(x_1,x_2,x_3)+f(y_1,y_2,y_3) \\ f(cx_1, cx_2, cx_3) &= cx_1+cx_2+cx_3 \\ &= c(x_1+x_2+x_3) \\ &= c\cdot f(x_1,x_2,x_3) \end{aligned}$$

Therefore, the map f is linear.

□

Problem 2. Consider the map $R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$R_\theta(x_1, x_2) := (\cos\theta \cdot x_1 + \sin\theta \cdot x_2, -\sin\theta \cdot x_1 + \cos\theta \cdot x_2),$$

where $\theta \in \mathbb{R}$ is a fixed angle. Each item is worth 5 points. Solve the following parts:

(1) Show that the map R_θ is linear.

(2) What are the values of $R_\theta(1,0)$,$R_\theta(0,1)$ and $R_\theta(1,1)$?

(3) Show that the map is injective.

(4) Show that the map is surjective.

(5) In words, and possibly using a picture, describe what the map R_θ is doing *geometrically* when applied to points (x_1, x_2) in the plane $\mathbb{R}^2$.

(6) (*Optional, extra 5 points*) Show that the composition $R_\theta \circ R_{-\theta}$ is the identity map, i.e. $(R_\theta \circ R_{-\theta})(x_1, x_2) = (x_1, x_2)$. (This is also true for $R_{-\theta} \circ R_\theta$.)

Solution. Recall the trigonometric identity, $\cos^2\theta + \sin^2\theta = 1$, which will be used throughout this problem.

(1) We show linearity by checking that R_θ satisfies vector addition and scalar multiplication.

$$\begin{aligned} R_\theta(x_1+y_1, x_2+y_2) &= (\cos\theta\cdot(x_1+y_1)+\sin\theta\cdot(x_2+y_2), -\sin\theta\cdot(x_1+y_1)+\cos\theta\cdot(x_2+y_2)) \\ &= (\cos\theta\cdot x_1+\cos\theta\cdot y_1+\sin\theta\cdot x_2+\sin\theta\cdot y_2, \\ &\qquad -\sin\theta\cdot x_1-\sin\theta\cdot y_1+\cos\theta\cdot x_2+\cos\theta\cdot y_2) \\ &= (\cos\theta\cdot x_1+\sin\theta\cdot x_2, -\sin\theta\cdot x_1+\cos\theta\cdot x_2) \\ &\quad + (\cos\theta\cdot y_1+\sin\theta\cdot y_2, -\sin\theta\cdot y_1+\cos\theta\cdot y_2) \\ &= R_\theta(x_1,x_2)+R_\theta(y_1,y_2) \\ R_\theta(cx_1, cx_2) &= (\cos\theta\cdot cx_1+\sin\theta\cdot cx_2, -\sin\theta\cdot cx_1+\cos\theta\cdot cx_2) \\ &= (c\cdot(\cos\theta\cdot x_1+\sin\theta\cdot x_2), c\cdot(-\sin\theta\cdot x_1+\cos\theta\cdot x_2)) \\ &= c\cdot(\cos\theta\cdot x_1+\sin\theta\cdot x_2, -\sin\theta\cdot x_1+\cos\theta\cdot x_2) \\ &= c\cdot R_\theta(x_1,x_2) \end{aligned}$$

Hence, R_θ is linear.

(2) We evaluate the map R_θ at the given point (x_1, x_2). For part (5) of this question it is useful to consider (x_1, x_2) as a vector from $(0, 0)$ to (x_1, x_2).

$$\begin{aligned} R_\theta(1,0) &= (\cos\theta\cdot 1+\sin\theta\cdot 0, -\sin\theta\cdot 1+\cos\theta\cdot 0) \\ &= \boxed{(\cos\theta, -\sin\theta)} \\ R_\theta(0,1) &= (\cos\theta\cdot 0+\sin\theta\cdot 1, -\sin\theta\cdot 0+\cos\theta\cdot 1) \\ &= \boxed{(\sin\theta, \cos\theta)} \\ R_\theta(1,1) &= (\cos\theta\cdot 1+\sin\theta\cdot 1, -\sin\theta\cdot 1+\cos\theta\cdot 1) \\ &= \boxed{(\cos\theta+\sin\theta, -\sin\theta+\cos\theta)} \end{aligned}$$

(3) To show that the map $\mathbb{R}_\theta$ is injective, we want to show that if $(x_1, x_2), (x_1', x_2') \in \mathbb{R}^2$ and $R_\theta(x_1, x_2) = R_\theta(x_1', x_2')$, then $(x_1, x_2) = (x_1', x_2')$.

Suppose that $R_\theta(x_1, x_2) = R_\theta(x_1', x_2')$, then we have that

$$\cos\theta\cdot x_1+\sin\theta\cdot x_2 = \cos\theta\cdot x_1'+\sin\theta\cdot x_2' \tag{1}$$
$$-\sin\theta\cdot x_1+\cos\theta\cdot x_2 = -\sin\theta\cdot x_1'+\cos\theta\cdot x_2' \tag{2}$$

Now, we want to show that $x_1 = x_1'$ and $x_2 = x_2'$. Note that since θ is fixed, then $\cos\theta$ and $\sin\theta$ are constants. First, we add $\sin\theta$ times equation (1) to $\cos\theta$ times equation (2) to get

$$\begin{aligned} (\sin^2\theta+\cos^2\theta)\cdot x_2 &= (\sin^2\theta+\cos^2\theta)\cdot x_2' \\ x_2 &= x_2'. \end{aligned}$$

Now we add $-\cos\theta$ times equation (1) to $\sin\theta$ times equation (2) to get

$$\begin{aligned} (-\cos^2\theta-\sin^2\theta)\cdot x_1 &= (-\cos^2\theta-\sin^2\theta)\cdot x_1' \\ x_1 &= x_1'. \end{aligned}$$

Therefore, we have shown that if $R_\theta(x_1, x_2) = R_\theta(x_1', x_2')$, then $(x_1, x_2) = (x_1', x_2')$, proving that the map R_θ is injective.

(4) To show that the map R_θ is surjective, we want to show that for all $(y_1, y_2) \in \mathbb{R}^2$ there exists an $(x_1, x_2) \in \mathbb{R}^2$ where $R_\theta(x_1, x_2) = (y_1, y_2)$, i.e., we want to solve the system of equations

$$\begin{cases} \cos\theta \cdot x_1 + \sin\theta \cdot x_2 = y_1 \\ -\sin\theta \cdot x_1 + \cos\theta \cdot x_2 = y_2 \end{cases}$$

First, add $\sin\theta$ times equation (1) to $\cos\theta$ times equation (2) and get

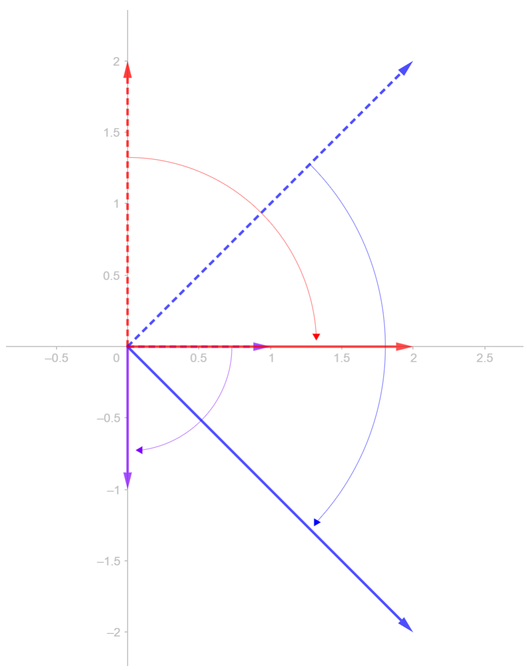
$$\begin{aligned} (\sin^2\theta + \cos^2\theta) \cdot x_2 &= y_1 \sin\theta + y_2 \cos\theta \\ x_2 &= y_1 \sin\theta + y_2 \cos\theta \end{aligned}$$

Next, we add $-\cos\theta$ times equation (1) to $\sin\theta$ times equation (2) and get

$$\begin{aligned} (-\cos^2\theta - \sin^2\theta) \cdot x_1 &= -y_1 \cos\theta + y_2 \sin\theta \\ -x_1 &= -y_1 \cos\theta + y_2 \sin\theta \\ x_1 &= y_1 \cos\theta - y_2 \sin\theta \end{aligned}$$

Therefore, we have shown that for all $(y_1, y_2) \in \mathbb{R}^2$ there exists $(x_1, x_2) = (y_1 \cos\theta - y_2 \sin\theta, y_1 \sin\theta + y_2 \cos\theta)$ such that $R_\theta(x_1, x_2) = (y_1, y_2)$; therefore, the map R_θ is surjective.

(5) The map R_θ acts on $\mathbb{R}^2$ by clockwise rotations of the angle θ. In the figure below, we have chosen $\theta = \frac{\pi}{2}$. With initial vectors $(1, 0), (0, 2), (2, 2)$ drawn as dashed line vectors and the resulting vectors under the map R_θ are given by solid line vectors. We can see that each vector through the origin has been rotated clockwise by an angle of $\frac{\pi}{2}$.



Fun fact: There is also a map which acts on $\mathbb{R}^2$ by counterclockwise rotations which is given by $R'_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ where

$$R'_\theta(x_1, x_2) = (\cos\theta \cdot x_1 - \sin\theta \cdot x_2, \sin\theta \cdot x_1 + \cos\theta \cdot x_2).$$

(6) (*Optional, extra 5 points*)
By direct computation we show that $(R_\theta \circ R_{-\theta})(x_1, x_2) = (x_1, x_2)$. We will use the fact that $\cos x$ is an even function, i.e., $\cos(-x) = \cos(x)$, and $\sin x$ is an odd function, i.e., $\sin(-x) = -\sin(x)$.

$$
\begin{aligned}
(R_\theta \circ \mathbb{R}_{-\theta})\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= R_\theta(R_{-\theta}(x_1, x_2)) \\
&= R_\theta \begin{pmatrix} \cos(-\theta)\cdot x_1 + \sin(-\theta)\cdot x_2 \\ -\sin(-\theta)\cdot x_1 + \cos(-\theta)\cdot x_2 \end{pmatrix} \\
&= R_\theta \begin{pmatrix} \cos\theta\cdot x_1 - \sin\theta\cdot x_2 \\ \sin\theta\cdot x_1 + \cos\theta\cdot x_2 \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta\cdot(\cos\theta\cdot x_1 - \sin\theta\cdot x_2) + \sin\theta\cdot(\sin\theta\cdot x_1 + \cos\theta\cdot x_2) \\ -\sin\theta\cdot(\cos\theta\cdot x_1 - \sin\theta\cdot x_2) + \cos\theta\cdot(\sin\theta\cdot x_1 + \cos\theta\cdot x_2) \end{pmatrix} \\
&= \begin{pmatrix} \cos^2\theta\cdot x_1 - \cos\theta\sin\theta\cdot x_2 + \sin^2\theta\cdot x_1 + \sin\theta\cos\theta\cdot x_2 \\ -\sin\theta\cos\theta\cdot x_1 + \sin^2\theta\cdot x_2 + \cos\theta\sin\theta\cdot x_1 + \cos^2\theta\cdot x_2 \end{pmatrix} \\
&= \begin{pmatrix} (\cos^2\theta + \sin^2\theta)\cdot x_1 \\ (\sin^2\theta + \cos^2\theta)\cdot x_2 \end{pmatrix} \\
&= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\end{aligned}
$$

A similar process holds for $(R_{-\theta} \circ R_\theta)(x_1, x_2)$. This identity makes sense because the map R_θ rotates vectors clockwise by an angle θ, whereas the map $R_{-\theta}$ rotates the vectors clockwise by an angle of $-\theta$ which is equivalent to rotating vectors counterclockwise by an angle of θ.

□

Problem 3. Consider the two maps

$$\pi_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2, \quad \pi_1(x_1, x_2, x_3) := (x_1, x_2 + 4x_3), \text{ and}$$
$$\pi_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3, \quad \pi_2(q_1, q_2) := (q_1 + 2q_2, q_2, q_1 - q_2).$$

Consider the composition $f = \pi_2 \circ \pi_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$. Each item is worth 5 points. Solve the following parts:

(1) Show that f is linear.

(2) What are the values of $f(1,0,0)$,$f(0,1,0)$ and $f(0,0,1)$?

(3) Is there any non-zero point $(a,b,c) \in \mathbb{R}^3$ such that $f(a,b,c) = (0,0,0)$?

(4) Show that the map f is *not* injective.

(5) Show that the map f is *not* surjective.

Solution.

(1) First, we determine the composition map $f = \pi_2 \circ \pi_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$.

$$\begin{aligned}(\pi_2 \circ \pi_1)(x_1, x_2, x_3) &= \pi_2(\pi_1(x_1, x_2, x_3) \\ &= \pi_2(x_1, x_2 + 4x_3)) \\ &= (x_1 + 2(x_2 + 4x_3), x_2 + 4x_3, x_1 - (x_2 + 4x_3)) \\ &= (x_1 + 2x_2 + 8x_3, x_2 + 4x_3, x_1 - x_2 - 4x_3)\end{aligned}$$

We now show that this map is linear by showing that vector addition and scalar multiplicity hold.

$$\begin{aligned}f(x_1+y_1, x_2+y_2, x_3+y_3) &= ((x_1+y_1) + 2(x_2+y_2) + 8(x_3+y_3), (x_2+y_2) + 4(x_3+y_3), \\ &\quad (x_1+y_1) - (x_2+y_2) - 4(x_3+y_3)) \\ &= (x_1 + y_1 + 2x_2 + 2y_2 + 8x_3 + 8y_3, x_2 + y_2 + 4x_3 + 4y_3, \\ &\quad x_1 + y_1 - x_2 - y_2 - 4x_3 - 4y_3) \\ &= (x_1 + 2x_2 + 8x_3, x_2 + 4x_3, x_1 - x_2 - 4x_3) \\ &\quad + (y_1 + 2y_2 + 8y_3, y_2 + 4y_3, y_1 - y_2 - 4y_3) \\ &= f(x_1, x_2, x_3) + f(y_1, y_2, y_3) \\ f(cx_1, cx_2, cx_3) &= (cx_1 + 2cx_2 + 8cx_3, cx_2 + 4cx_3, cx_1 - cx_2 - 4cx_3) \\ &= (c(x_1 + 2x_2 + 8x_3), c(x_2 + 4x_3), c(x_1 - x_2 - 4x_3)) \\ &= c \cdot (x_1 + 2x_2 + 8x_3, x_2 + 4x_3, x_1 - x_2 - 4x_3) \\ &= c \cdot f(x_1, x_2, x_3)\end{aligned}$$

Therefore, the composition map $f = \pi_2 \circ \pi_1$ is linear.

(2) We evaluate the composition map f at the indicated points.

$$f(1,0,0) = (1 + 2(0) + 8(0), 0 + 4(0), 1 - 0 - 4(0)) = \boxed{(1,0,1)},$$

$$f(0,1,0) = (0 + 2(1) + 8(0), 1 + 4(0), 0 - 1 - 4(0)) = \boxed{(2,1,-1)},$$

$$f(0,0,1) = (0 + 2(0) + 8(1), 0 + 4(1), 0 - 0 - 4(1)) = \boxed{(8,4,-4)}.$$

(3) To determine which points map to $(0,0,0)$ under the map f we solve the following system of equations:

$$\begin{cases} x_1 + 2x_2 + 8x_3 = 0 \\ x_2 + 4x_3 = 0 \\ x_1 - x_2 - 4x_3 = 0 \end{cases}$$

From equation (2) we see that we may solve for x_2 and we find that $x_2 = -4x_3$. By substituting x_2 into equation (1) we find that $x_1 = 0$. Also, substituting x_2 into equation (3) we also find that $x_1 = 0$. However, there is no other equation we may use to further solve for x_3; therefore, there are infinitely many solutions of the form $(0, -4x_3, x_3)$.

(4) Recall that a function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is injective if $f(x_1, x_2, x_3) = f(x_1', x_2', x_3')$ then $(x_1, x_2, x_3) = (x_1', x_2', x_3')$. To show that a map is not injective it suffices to

find a counterexample which demonstrates that there exists two distinct points in the domain which map to the same point under the map f.

We choose two points $(0, -4, 1)$ and $(0, 8, -2)$ and compute their image under the map f,

$$f(0,-4,1) = (0 + 2(-4) + 8(1), -4 + 4(1), 0 - (-4) - 4(1)) = (0,0,0),$$
$$f(0,8,-2) = (0 + 2(8) + 8(-2), 8 + 4(-2), 0 - 8 - 4(-2)) = (0,0,0).$$

Therefore, $f(0,-4,1) = f(0,8,-2) = (0,0,0)$ but $(0,-4,1) \neq (0,8,-2)$; hence, the map is not injective.

(5) A function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is surjective if for all $(y_1, y_2, y_3) \in \mathbb{R}^3$ there exists $(x_1, x_2, x_3) \in \mathbb{R}^3$ such that $f(x_1, x_2, x_3) = (y_1, y_2, y_3)$. So to show a map f is not surjective it suffices to find a point in the codomain such that there does not exist an element in the domain which maps to it, i.e., there are no solutions to the systems of equations that the map f represents.

Let $y = (1, 0, 0)$, the goal is to show that there are no solutions to the following system of equations

$$\begin{cases} x_1 + 2x_2 + 8x_3 = 1 \\ x_2 + 4x_3 = 0 \\ x_1 - x_2 - 4x_3 = 0 \end{cases}$$

We solve for x_2 in equation (2) to find that $x_2 = -4x_3$. Now, if we plug this result into equation (1) we see that $x_1 = 1$. However, if we plug the value of x_2 into equation (3) we see that $x_1 = 0$. Since $1 \neq 0$, we have that this system of equations has no solutions, i.e., there is no point (x_1, x_2, x_3) such that $f(x_1, x_2, x_3) = (1, 0, 0)$. Hence, f is not surjective.

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Problem 4. Solve each of the following parts. Each item is worth 5 points:

(1) Find a linear map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $f(1,0) = (4,5)$ and $f(0,1) = (3,-2)$.

(2) Suppose that $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear map such that $f(1,0) = (4,5)$ and $f(0,1) = (3,-2)$. Find $f(5,7)$.

(3) Find two distinct linear maps $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $f(1,0) = (4,5)$ and $g(1,0) = (4,5)$.

(4) Show that there is no linear map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $f(1,1) = (2,-3)$ and $f(0,1) = (5,6)$ and $f(2,1) = (10,4)$.

(5) Is there a linear map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $f(1,0) = (0,0)$ and f is surjective?

Solution.

(1) All linear maps $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are of the form $f(x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2)$, first we evaluate the map at the given points and use this data to explicitly find the function f.

$$f(1,0) = (a(1) + b(0), c(1) + d(0)) = (a, c) = (4, 5),$$
$$f(0,1) = (a(0) + b(1), c(0) + d(1)) = (b, d) = (3, -2).$$

Therefore, the desired linear map f is defined $f(x_1, x_2) = (4x_1 + 3x_2, 5x_1 - 2x_2)$.

(2) We may have used the equation found in part (1). However, here we use the properties linearity of the map to compute $f(5, 7)$. Since

$$(5,7) = 5(1,0) + 7(0,1),$$

we see that

$$f(5,7) = 5f(1,0) + 7f(0,1) = 5(4,5) + 7(3,-2)$$
$$= (20,25) + (21,-14) = \boxed{(41,11)}.$$

(3) Given that we want to find a map $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that is linear, then it must be of the form $g(x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2)$. By computing g at the given point $(1, 0)$ we see that

$$g(1,0) = (a, c) = (4, 5),$$

therefore, we know that $g(x_1, x_2) = (4x_1 + bx_2, 5x_1 + dx_2)$. Since we are not given any more constraining data points we can choose $b, d \in \mathbb{R}$ to be any values, i.e., there are infinitely many solutions. However, we are looking for a linear map that is different than the map f found in part (1) of this problem; therefore, $b \neq 3$ and $d \neq -2$. So for example, let $b = 4, d = 2024$ and then

$$\boxed{g(x_1, x_2) = (4x_1 + 4x_2, 5x_3 + 2024x_2).}$$

(4) Since $(2, 1) = 2(1, 1) - (0, 1)$, then

$$f(2,1) = 2f(1,1) - f(0,1) = 2(2,-3) - (5,6)$$
$$= (4,-6) - (5,6) = (-1,-12)$$

using the linearity properties of the map f. However, the given value is $(10, 4) \neq (-1, 0)$; therefore, there is no linear map which satisfies the given points.

(5) $\boxed{No}$, it is not possible to have a linear map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $f(1, 0) = (0, 0)$ and f is surjective. Recall that a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$ is of the form $f(x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2)$ and using the condition that $f(1, 0) = (0, 0)$, the map f is defined

$$f(x_1, x_2) = (bx_2, dx_2)$$

for some real numbers $b, d \in \mathbb{R}$. Since b, d are fixed numbers, then the image of the map is a line in $\mathbb{R}^2$ through the origin with slope $\frac{d}{b}$. For example, if $b = 2, d = 3$, then the linear map is defined $f(x_1, x_2) = (2x_2, 3x_2) = x_2(2, 3)$. Here, x_2 acts as the scalar multiple of the vector $(2, 3)$.

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