Creed Alexander Erickson IV
Department of Mathematics, University of California, Davis.
E-mail: `caverick@math.ucdavis.edu`
Website: `http://www.math.ucdavis.edu/∼caverick/`

June 28, 2008

Completion of metric spaces:
Problem from **Principles of Mathematical Analysis** by Walter Rudin 3'rd ed.

Exercise 24 Chapter 3 Let X be a metric space

(a) Call two Cauchy sequences $(p_n), (q_n)$ in X equivalent if

$$\lim_{n\to\infty} d(p_n, q_n) = 0$$

Prove that this is an equivalence relation.

(b) Let X^* be the set of all equivalence classes so obtained. If $P \in X^*, Q \in X^*, (p_n) \in P$ and $(q_n) \in Q$ define

$$\triangle(P, Q) = \lim_{n\to\infty} d(p_n, q_n);$$

by Exercise 23, this limit exists. Show that the number $\triangle(P, Q)$ is unchanged if (p_n) and (q_n) are replaced by equivalent sequences, and hence that $\triangle$ is a distance function in X^*

(c) Prove that the resulting meric space X^* is complete

(d) For each $p \in X$, there is a Cauchy sequence all of whose terms are p; let P_p be the element of X^* which contains this sequence. Prove that

$$\triangle(P_p, P_q) = d(p, q)$$

for all $p, q \in X$. In other words, the mapping $\varphi(p) = P_p$ is an isometry of X into X^*.

(e) Prove that $\varphi(X)$ is dense in X^*, and that $\varphi(X) = X^*$ if X is complete. By (d), we may identify X and $\varphi(X)$ and thus regard X as embedded in the complete metric space X^*. We call X^* the `completion` of X.

Solution:

(a) Recall that an equivalence relation is a relation $\sim$ on a set X such that

$$\forall x \in X \;\; x \sim x$$
$$\forall x, y \in X \;\; x \sim y \Rightarrow y \sim x$$
$$\forall x, y, z \in X \;\; x \sim y \;\;\&\;\; y \sim z \Rightarrow x \sim z$$

Clearly for any Cauchy sequence (p_n) in X, $(p_n) \sim (p_n)$, since $d(p_n, p_n) = 0 \; \forall n$.
$(p_n) \sim (q_n) \Rightarrow (q_n) \sim (p_n)$ via symmetry of the metric d.
Assume $\lim_{n\to\infty} d(p_n, q_n) = 0$ and $\lim_{n\to\infty} d(q_n, r_n) = 0$ for Cauchy sequences $(p_n), (q_n), \&(r_n)$. Then since $d(p_n, r_n) \le d(p_n, q_n) + (q_n, r_n)$, we can take the limit as n goes to infinity on each side and see that $\lim_{n\to\infty} d(p_n, r_n) = 0$, so $(p_n) \sim (r_n)$

(b) Let $(p_n), (p'_n) \in P$ and $(q_n) \in Q$, i.e. (p_n) and (p'_n) are equivalent Cauchy sequences. Since $d(p'_n, q_n) \le d(p'_n, p_n) + d(p_n, q_n)$ and $d(p_n, q_n) \le d(p_n, p'_n) + d(p'_n, q_n)$ taking limits we get $\lim_{n\to\infty} d(p'_n, q_n) \le \lim_{n\to\infty} d(p_n, q_n)$ and $\lim_{n\to\infty} d(p_n, q_n) \le \lim_{n\to\infty} d(p'_n, q_n)$ this shows that $\triangle(P, Q)$ does not change if we replace (p_n) by an equivalent Cauchy sequence. We can also replace the Cauchy sequence for Q and using triangle inequality, $d(p'_n, q'_n) \le d(p'_n, q_n) + d(q_n, q'_n)$, and $d(p'_n, q_n) \le d(p'_n, q'_n) + d(q'_n, q_n)$ we see that $\triangle$ is a well defined distance function on X^*.

(c) Note that $\triangle$ does define a metric since it is well-defined and the required metric properties easily follow from those of d. So we only need to show that Cauchy sequences converge in X^*. First we need to identify a point of convergence for an arbitrary Cauchy sequence in X^*. Let (P_n) be Cauchy in X^*, i.e. $\forall \epsilon > 0$, $\exists$ N s.t. $\triangle(P_n, P_m) < \epsilon$, $\forall n, m > N$. But $\forall n$, $\exists (x_k^n)$ Cauchy in X via k s.t. $[(x_k^n)] = P_n$, i.e. (x_k^n) is a representative of the equivalence class P_n. Consider the diagonal sequence (x_k^k), this is Cauchy since $d(x_k^k, x_l^l) \leq d(x_k^k, x_l^k) + d(x_l^k, x_k^l) + d(x_k^l, x_l^l)$, and the first and last terms are as small as desired since (x_m^k) and (x_n^l) are both Cauchy in m and n respectively. The middle term is $d(x_l^k, x_k^l) = \triangle(P_k, P_l)$ which is as small as desired by assumption. Call that diagonal Cauchy sequence P'. Now that we have a candidate element in X^*, we must show that (P_n) actually converges to this element. Explicitly, we need to show that $\forall \epsilon > 0, \exists N$ s.t. $n > N \Rightarrow \triangle(P', P_n) < \epsilon$. But from definition $\triangle(P', P_n) = \lim_{k\to\infty} d(p'_k, p_k^n)$, where $P' = [(p'_k)]$ and $P = [(p_k^n)]$. But from construction $[(p'_k)] = [(x_k^k)]$ and $[(p_k^n)] = [(x_k^n)]$, and $d(x_k^k, x_k^n) \leq d(x_k^k, x_l^k) + d(x_l^k, d_l^n) + d(x_l^n, x_k^n)$. As before, the first and last terms are as small as desired since (x_m^k) and (x_l^n) are both Cauchy in m and l respectively. The middle term is $d(x_l^k, x_k^n) = \triangle(P_k, P_n)$ which is as small as desired since P_m is Cauchy.

(d) Given $p, q \in X$, let $P_p = (p_n)$ where $\forall n, p_n = p$, likewise for P_q. Then $\triangle(P_p, P_q) = \lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(p, q) = d(p, q)$

(e) Let P be an arbitrary element in X^*. By definition there exists a Cauchy sequence (x_n) in X s.t. $[(x_n)] = P$. Now for all n, we can consider x_n as being in X^* via creating a constant Cauchy sequence out of x_n. Now since $\lim_{n\to\infty} \triangle(P, x_n) = \lim_{n\to\infty} \lim_{m\to\infty} d(x_m, x_n) = 0$, since (x_n) is Cauchy. So since $x_n \in \varphi(X)$, we have $\varphi(X)$ is dense in X^*. If X is complete then all Cauchy sequences in X converge, and any $P \in X^*$ is the image of the constant sequence which consists only of the convergent point to which all the equivalent Cauchy sequences converge too.