4-23
Comments on HW# 4:
In the $\log\log\left(1+\frac{1}{|x|}\right)$ problem let $n=2$for simplification.
In the problem with difference quotients, we may use any subset of $\mathbb{R}^n$. If $\Omega\subset\mathbb{R}^n$ is our choice, we need $\omega=\{x\in\Omega|d(x,\partial\Omega)\leq\epsilon\}$ to leave room for differencing. Then $\|D^n u\|_{L^1(\omega)}<M$ does not imply that $\|Du\|_{L^2(\omega)}$ is bounded.
In the interpolation problem, we can write
$\|Du\|^2_{L^2(\mathbb{R}^n)}\leq C\|u\|_{L^2(\mathbb{R}^n)}\|D^2u\|_{L^2(\mathbb{R}^n)}$
and this get realy interesting when we replace $\mathbb{R}^n$ with Ω.

We left off proving the Gagliardo-Nirenberg inequalty: If 1 ≤p<n and $u\in W^{1,p}(\mathbb{R}^n)$ then $u\in L^{p^*}(\mathbb{R}^n)$.
It suffices to show $\|u\|_{L^{p^*}(\mathbb{R}^n)}\leq C\|Du\|_{L^p(\mathbb{R}^n)}$ with $p^*=\dfrac{np}{n-p}$ (The Sobolev conjugate) and with C depending only on n and p. We where attempting the proof for $n=2$ since it is prototypical of the $n\geq 2$ cases and the $n=1$ case is boring. We further took the particluar case $p=1,p^*=2$ and using the fundamental theorem of calculus we showed that
$\|v\|_{L^2(\mathbb{R}^2)}\leq C\|Dv\|_{L^1(\mathbb{R}^2)}$
Part 2 of the proof: for 1<p<2 substitute $v=|u|^\gamma$ into $\|v\|_{L^2(\mathbb{R}^2)}\leq C\|Dv\|_{L^1(\mathbb{R}^2)}$

$$\left(\int_{\mathbb{R}^2}|u|^{2\gamma}\,dx\right)^{1/2}\leq c|\gamma|\int_{\mathbb{R}^2}|u|^{\gamma-1}|Du|\,dx$$

and then by Holder's inequality

$$\leq C\|Du\|_{L^p(\mathbb{R}^2)}\left(\int_{\mathbb{R}^2}|u|^{(\gamma-1)\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}}$$

We want for $\|u\|_{L^{\frac{2p}{2-p}}(\mathbb{R}^2)}\leq C\|Du\|_{L^p(\mathbb{R}^2)}$ so we set $\gamma=\dfrac{p}{2-p}>1$ so we get

$$\left(\int_{\mathbb{R}^2}|u|^{\frac{2p}{2-p}}\,dx\right)^{1/2}\leq C\|Du\|_{L^p(\mathbb{R}^2)}\left(\int_{\mathbb{R}^2}|u|^{\frac{2p}{2-p}}\,dx\right)^{\frac{p-1}{p}}$$

This is not of the form $\|u\|_{L^{p^*}(\mathbb{R}^n)}\leq C\|Du\|_{L^p(\mathbb{R}^n)}$, but proves the embedding nonetheless.

So what is this all good for? Where are we at?

Sobolev spaces on bounded domains:
We will see two interesting cases
Case 1: $\Omega\subset\mathbb{R}^n$ bounded with smooth boundary (from now on by the word bounded we mean $\mu(\Omega)<\infty$ and $\partial\Omega$ smooth)
Case 2: $\Omega=T^n$.
recal our Dfn: $W^{k,p}(\Omega)=\{u\in L^p(\Omega)\,|D^\alpha u\in L^p(\Omega),|\alpha|\leq k\}$ where $k\in\mathbb{N}$

and $1 \leq p \leq \infty$ and $\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\leq k} \|D^{\alpha}u\|_{L^p}^p\right)^{1/p}$ or equivalently $\|u\|_{W^{k,p}(\Omega)} = \sum_{|\alpha|\leq k} \|D^{\alpha}u\|_{L^p}$.

The two interesting subspaces of $W^{1,p}(\Omega)$ are

1) $W_0^{1,p}(\Omega)$ =the closure of $C_0^{\infty}(\Omega)$ with respect to the norm $\|\cdot\|_{W^{1,p}(\Omega)}$. Ie the space of functiomns which are limits of sequences of smooth functions with compact support with respect to the $\|\cdot\|_{W^{1,p}(\Omega)}$ norm.

2) $W^{1,p}(T^n)$ =periodic functions on $\mathbb{R}^n$ with $W^{1,p}(T^n)$ regularity. ?24? what does this regularity reffer to?

The interseting subspace is $\dot{W}^{1,p}(T^n) = \left\{u \in W^{1,p}(T^n) \mid \int_{T^n} u(x)\,dx = 0\right\}$ ie, non-constant functions (zeroth Fourier mode is 0)

Some corrolaries of the Gagliardo-Nirenberg inequality:

Theorem 1 (estimates for $W^{1,p}(\Omega)$): If Ω is bounded and $1 \leq p < n$ and $u \in W^{1,p}(\Omega)$ then $\|u\|_{L^{p^*}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$ with $C = C(p,n,\Omega)$ and $p^* = \dfrac{np}{n-p}$.

Theorem 2: If $\Omega \subset \mathbb{R}^n$ bounded and $u \in W_0^{1,p}(\Omega)$ and $1 \leq p < n$ then $u \in L^q(\Omega)$ for all $q \in [1, p^*]$ and $\|u\|_{L^q(\Omega)} \leq C\|Du\|_{L^p(\Omega)}$

Example: $n = 3$ and $p = 2$ so that $p^* = 6$. Thus $q = 2$ works in 3-d:

$\|u\|_{L^2(\Omega)} \leq C\|Du\|_{L^2(\Omega)}$ for all $u \in W_0^{1,2}(\Omega) = H_0^1(\Omega)$

This last equality is called the Poincare inequality. It has generalizations we will look at later in the course.

Remark: For all $u \in H_0^1(\Omega)$ $\|Du\|_{L^2(\Omega)}$ is an $H^1(\Omega)$ equivalent norm. (ie there exist constants c_1, c_2 such that $c_1\|u\|_{H^1(\Omega)} \leq \|Du\|_{L^2(\Omega)} \leq c_2\|u\|_{H^1(\Omega)}$, where the right inequality is trivial to prove, and the left is Poincare inequality, and this is what makes the inequality so interesting.)

Poisson's equation on $\Omega \subset \mathbb{R}^3$: $-\Delta u = \partial_i\partial^i u = f$ with boundary condition $u = 0$ on $\partial\Omega$.

Definition: $u \in H_0^1(\Omega)$ is a weak solution to $-\Delta u = f$ if

$\int_{\Omega}(Du)\cdot(Dv)\,dx = \int_{\Omega} fv dx$ for all $v \in C_0^{\infty}(\Omega)$

Motivation: Dont think of $-\Delta u = f$ pointwise anymore. Multiply both sides by a test function ϕ and integrate

$$-\int_{\Omega}\text{Div}(Du)\,\phi dx = \int_{\Omega} f\phi dx$$

integrate by parts

$\int_{\Omega} f\phi dx = \int_{\Omega}(Du)\cdot(D\phi)\,dx$ with the boundary term equal to zero. So this is our equation of interest, and it is not a pointwise relationship!

end of lecture.