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Last time we defined $S\left(\mathbb{R}^n\right)$, an alternative definition is
$S\left(\mathbb{R}^n\right)=\left\{u\in C^{\infty}\left(\mathbb{R}^n\right)|\exists C_{k,\alpha}\boldsymbol{s.t.}\,\langle x\rangle^k\left|D^{\alpha}u\left(x\right)\right|\leq C_{k,\alpha}\forall k\geq 0,|\alpha|\geq 0\right\}$
where, as in previous lectures, $\langle x\rangle=\sqrt{1+|x|^2}$.
Dfn: For $k\in\mathbb{N}$ set $p_k\left(u\right)=\sup\limits_{x\in\mathbb{R}^n,|\alpha|\leq k}\langle x\rangle^k\left|D^{\alpha}u\left(x\right)\right|$. this is called the seminorm on $S\left(\mathbb{R}^n\right)$.
Dfn: $d\left(u,v\right)=\sum\limits_{k=0}^{\infty}2^{-k}\frac{p_k\left(u-v\right)}{1-p_k\left(u-v\right)}$ is the distance (aka metric) function on $S\left(\mathbb{R}^n\right)$.
Fact: $\left(S\left(\mathbb{R}^n\right),d\right)$ is a Fre'chet space (ie a complete metric space which is localy convex)
Dfn (topology of strong convergence (with respect to the metric)): We say that $u_j\rightarrow u$ in $S\left(\mathbb{R}^n\right)$ if $p_k\left(u_j-u\right)\rightarrow 0$ as $j\rightarrow\infty$ for all $k\in\mathbb{N}$.
Tempered Distributions:
Dfn: A linear map $W:S\left(\mathbb{R}^n\right)\rightarrow\mathbb{C}$ is continuous iff there exist k,C such that $\left|W\left(u\right)\right|\leq Cp_k\left(u\right)$ for all $u\in S\left(\mathbb{R}^n\right)$.
Dfn: $S'\left(\mathbb{R}^n\right)$ is the set of continuous linear functionals $\{W:S\left(\mathbb{R}^n\right)\rightarrow\mathbb{C}\}$.
Dfn (weak star topology on $S'\left(\mathbb{R}^n\right)$): We say W_j converges to W weak star in $S'\left(\mathbb{R}^n\right)$ if $W_j\left(u\right)\rightarrow W\left(u\right)$ for all $u\in S\left(\mathbb{R}^n\right)$.
Notation: The action of $W\in S'\left(\mathbb{R}^n\right)$ on $u\in S\left(\mathbb{R}^n\right)$ is often written as $W\left(u\right)=\langle u,W\rangle$.
Facts:
1)There is a natural injection $L^p\left(\mathbb{R}^n\right)\hookrightarrow S'\left(\mathbb{R}^n\right)$ for all $p\in[1,\infty]$ given by $\langle u,f\rangle=\int_{\left(\mathbb{R}^n\right)}u\left(x\right)f\left(x\right)dx$.
2) Any finite measure on $\mathbb{R}^n$ gives an element of $S'\left(\mathbb{R}^n\right)$. The basic example of this is "the delta function" (at x) defined as $\langle u,\delta_x\rangle=u\left(0\right)$.
3) The partial derivative operator $\partial_j:S'\left(\mathbb{R}^n\right)\rightarrow S'\left(\mathbb{R}^n\right)$ is defined by $\langle u,\partial_jW\rangle=-\langle\partial_ju,W\rangle$ for all $u\in S\left(\mathbb{R}^n\right)$.
4) Multiplication of any $W\in S'\left(\mathbb{R}^n\right)$ by an element of $S\left(\mathbb{R}^n\right)$ is defined by $\langle u,fW\rangle=\langle fu,W\rangle$.
Examples: The Heaviside function
$\eta\left(x\right)=\begin{cases}1\text{ if }x\geq 0\\0\text{ if }x<0\end{cases}$
Is in $L^{\infty}\left(\mathbb{R}^n\right)$, so by 1 above it gives a distribution. Lets compute $\frac{d}{dx}H\left(x\right)$.
Claim: $\frac{dH}{dx}=\delta_0$ in $S'\left(\mathbb{R}^n\right)$

Proof: $\left\langle u, \dfrac{d}{dx}H \right\rangle = -\left\langle \dfrac{du}{dx}, H \right\rangle = -\displaystyle\int_{\mathbb{R}} \dfrac{du}{dx} H dx = -\int_0^\infty \dfrac{du}{dx} dx = u\left(0\right)$
$= \left\langle u, \delta_0 \right\rangle$
Example: In $n = 1$ we have $\left\langle u, \dfrac{d\delta}{dx} \right\rangle = -\left\langle \dfrac{du}{dx}, \delta \right\rangle = -\dfrac{du}{dx}\left(0\right)$
We can now define $\mathcal{F}$ of a tempered distribution:
Motivation: What is $\mathcal{F}\left(1\right)$?
Extension of $\mathcal{F}$ to to $S'\left(\mathbb{R}^n\right)$:
Dfn: $\mathcal{F} : S'\left(\mathbb{R}^n\right) \to S'\left(\mathbb{R}^n\right)$ is defined implicitly by $\left\langle u, \mathcal{F}W \right\rangle = \left\langle \mathcal{F}u, W \right\rangle$.
(Similarly for $\mathcal{F}^*$)
We'd like to have the Fourier inversion formula extend to $S'\left(\mathbb{R}^n\right)$.
Thm: $\mathcal{F}^*\mathcal{F} = \mathcal{F}\mathcal{F}^* = id$ on $S'\left(\mathbb{R}^n\right)$.
Proof: $\left\langle u, \mathcal{F}^*\mathcal{F}W \right\rangle := \left\langle \mathcal{F}\mathcal{F}^*u, W \right\rangle = \left\langle u, W \right\rangle$ QED.
Example: $\mathcal{F}\delta_0 = \left(2\pi\right)^{-n/2}$ since $\left\langle u, \mathcal{F}\delta_0 \right\rangle = \left\langle \mathcal{F}u, \delta_0 \right\rangle = \left(\mathcal{F}u\right)\left(0\right)$
$= \left(2\pi\right)^{-n/2} \displaystyle\int_{\mathbb{R}^n} u\left(x\right) e^0 dx = \int_{\mathbb{R}^n} u\left(x\right) \left(2\pi\right)^{-n/2} dx = \left\langle u, \left(2\pi\right)^{-n/2} \right\rangle$
Where $\left(2\pi\right)^{-n/2}$ is to be thought of as the image of $\left(2\pi\right)^{-n/2}$ under the imbedding $L^\infty\left(\mathbb{R}^n\right) \hookrightarrow S'\left(\mathbb{R}^n\right)$.

End of lecture