Math 16A Short Calculus

Practice Final Exam

1. **Continuity**

   For each of the following functions, state the domain and the interval(s) on which the function is continuous.

   (a) $y = x^2 + 4$

   (b) $y = \frac{x+2}{x^2-4}$

   (c) $y = 2\lfloor x \rfloor$

   (d) $y = 4\cos(\theta) + 1$

2. **Slope and Derivative**

(a) Find an equation of the line tangent to $f(x) = x^2 + 1$ with slope -2.

(b) Use the limit definition to find the derivative of the following functions.

i. $f(x) = x^2 - 1$

ii. $h(t) = 2\sqrt{t}$

3. **Rates of Change**

(a) The profit P from selling x units of a product is given by

$$P = 15 + 12\sqrt{x} - \frac{81}{x}.$$

i. Find the average rate of change of P on the interval $[1, 9]$.

ii. Find a formula for the marginal profit. What is the marginal profit for $x = 9$?

(b) A sphere of radius 3 has its radius changing at a rate of 4 inches per second. How quickly is the volume of the sphere changing? (You may use the fact the volume of a sphere with radius r is $\frac{4}{3}\pi r^3$)

4. **Rules of Differentiation** (40 points)

(a) For each of the following functions, find the derivative. (30 points; 10 points each)

i. $f(x) = \frac{2\sqrt{x}+1}{\sqrt{x}}$

ii. $g(t) = t^3 + 3t^2 - 5$

iii. $h(x) = \frac{2x+1}{x+2}$

(b) At which values for x does the graph of $y = -x^3 + 9x$ have a horizontal tangent line? (10 points)

5. **Rates of Change** (30 points)

(a) The profit P from selling x units of a product is given by

$$P = 30 + 12\sqrt{x} - \frac{81}{x}.$$

Find a formula for the marginal profit. What is the marginal profit for $x = 9$? (10 points)

(b) Consider $f(x) = (x^2 - x + 2)(x^2 + x)$.

i. What is the average rate of change of $f(x)$ on the interval $[0, 1]$? (10 points)

ii. What is the instantaneous rate of change of $f(x)$ at $x = 0$, and at $x = 1$? (10 points)

6. **True/False** (30 points, 5 points each)
Mark each question as (**T**)rue or (**F**)alse.

a. ___ The derivative of $\sqrt{x}$ is always non-negative.

b. ___ π radians = 180 degrees.

c. ___ A derivative of a position function with respect to time is a velocity.

d. ___ To apply the Quotient Rule to $f(x)/g(x)$, we must have that $g(x)$ is always positive.

e. ___ $\cos(\pi - \theta) = \cos\theta$.

f. ___ If a function is continuous at a point, then it is differentiable at that point.

7. **Derivative Tests**

(a) Apply the Second-Derivative Test to find and classify the relative extrema of $f(x) = x^3 - 3x^2 + 1$. Show all of your work.

(b) Apply the First-Derivative Test to find and classify the relative extrema of $g(t) = t^4 - 4t^3 - 4$. Show all of your work.

8. **Optimization**

(a) Find two positive numbers such that their product is 75, and so that the first plus three times the second is a minimum.

(b) A rectangular page is to contain 18 square inches of print. The margins at the top and bottom are .5 inches and on each side are 1 inch. What dimensions minimize the amount of paper used?

9. **Sketching Graphs** (60 points)

Sketch the graphs of the following functions, showing all work. In particular you should: state the domain; find and label all asymptotes; find and show all intercepts; find and show relative extrema and points of inflection.

(a) $g(x) = \frac{x^2}{x^2-1}$

(continued on next page)

6. Sketching Graphs (continued)

(b) $f(x) = \frac{x^2-2}{x^2-x-2}$

10. **True/False**

Mark each question as (**T**)rue or (**F**)alse.

a. ___ The absolute maximum of a function on a closed interval never occurs at the endpoints.

b. ___ The derivative of $\sec(x)$ is $-\sec(x)\tan(x)$.

c. ___ A function can only change from increasing to decreasing at a critical number.

d. ___ A function cannot intersect its horizontal asymptotes.

e. ___ If $f'(x) > 0$ for all x, then f is concave upward for all x.

f. ___ The Product Rule states: $\frac{d}{dx}[f(x)g(x)] = g(x)f'(x) + f(x)g'(x)$.

g. ___ A point of inflection of f can occur *only* at a critical number of f'.

11. **Chain Rule** (40 points)

(a) The number of bacteria in a culture after t days is modeled by

$$N(t) = 400\left(1 - \frac{4}{(t^2+3)^2}\right).$$

How many bacteria are there when $t = 1$? What is the rate of change of the bacteria population when $t = 1$? (15 points)

(b) Find the equation of the tangent line to the graph of $f(x) = (9 - x^3)^{\frac{-1}{2}}$ when $x = 2$. (15 points)

(c) Compute the derivative of $f(t) = \sqrt{t^2+4} - \sqrt{t^2-4}$. (10 points)

12. **Trig Functions** (40 points)

(a) Show that, if $y = \frac{5-\cos x}{x}$, then $xy' + y = \sin x$. (10 points)

(b) Find the derivatives of the following functions. (30 points; 10 points each)

i. $y = x \tan(5x)$

ii. $T = x - 4\csc(\pi x + 1)$

iii. $f(t) = \sqrt{\sin^2 t - \cos^2 t}$

13. **Higher-Order Derivatives and Implicit Differentiation** (30 points)

(a) Find dy/dx if $\sec(xy) = 5$. (15 points)

(b) What is the rate of change of t with respect to y (*not* y with respect to t!) if $y = \sqrt{t+1}$? (15 points)

14. **Related Rates** (40 points)

(a) You are given that the instantaneous rate of change for x with respect to time is 5 when $x = 3$. If $y = 1 + 2xy$, what is the corresponding value of $\frac{dy}{dt}$? (15 points)

(b) The radius r of a sphere is increasing by 3 feet per minute.

i. What is the rate of change of the volume V when $r = 10$ feet? ($V = \frac{4}{3}\pi r^3$) (15 points)

ii. What is $\frac{dV}{dt}$ when $V = 36\pi$? (10 points)

15. **True/False** (20 points, 5 points each)
Mark each question as (**T**)rue or (**F**)alse.

a. ___ The surface area of a sphere of radius r is $\frac{4}{3}\pi r^2$.

b. ___ Acceleration is the second derivative of velocity.

c. ___ We used the Chain Rule to prove that $\frac{d}{dt}[sin(t)] = cos(t)$.

d. ___ $\csc(2t) = \frac{2}{sin(t)}$.