

Math 16A Short Calculus, section 2
Practice Final Exam Answers

1. Continuity

For each of the following functions, state the domain and the interval(s) on which the function is continuous.

(a) $y = x^2 + 4$

Domain: $(-\infty, \infty)$

Continuous on: $(-\infty, \infty)$

(b) $y = \frac{x+2}{x^2-4}$

Domain: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

Continuous on: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

(c) $y = 2[x]$

Domain: $(-\infty, \infty)$

Continuous on: $[x, x + 1)$ for all integers x

(d) $y = 4 \cos(\theta) + 1$

Domain: $(-\infty, \infty)$

Continuous on: $(-\infty, \infty)$

No discontinuities

2. Slope and Derivative

- (a) Find an equation of the line tangent to $f(x) = x^2 + 1$ with slope -2 .

We have $f'(x) = 2x$, so slope -2 occurs when $-2 = f'(x) = 2x \implies x = -1$.
When $x = -1$, $f(-1) = 2$. The tangent line is:

$$(y - 2) = (-2)(x + 1) \implies y = -2x.$$

- (b) Use the limit definition to find the derivative of the following functions.

i. $f(x) = x^2 - 1$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - 1 - x^2 + 1}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{(\Delta x)(2x + \Delta x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} (2x + \Delta x) = 2x. \end{aligned}$$

ii. $h(t) = 2\sqrt{t}$

$$\begin{aligned} h'(x) &= \lim_{\Delta t \rightarrow 0} \frac{h(t + \Delta t) - h(t)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{2\sqrt{t + \Delta t} - 2\sqrt{t}}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{2\sqrt{t + \Delta t} - 2\sqrt{t}}{\Delta t} \cdot \frac{\sqrt{t + \Delta t} + \sqrt{t}}{\sqrt{t + \Delta t} + \sqrt{t}} \\ &= \lim_{\Delta t \rightarrow 0} \frac{2(t + \Delta t) - 2t}{(\Delta t)(\sqrt{t + \Delta t} + \sqrt{t})} \\ &= \lim_{\Delta t \rightarrow 0} \frac{2\Delta t}{(\Delta t)(\sqrt{t + \Delta t} + \sqrt{t})} \\ &= \lim_{\Delta t \rightarrow 0} \frac{2}{\sqrt{t + \Delta t} + \sqrt{t}} = \frac{2}{2\sqrt{t}} = \frac{1}{\sqrt{t}}. \end{aligned}$$

3. Rates of Change

(a) The profit P from selling x units of a product is given by

$$P = 15 + 12\sqrt{x} - \frac{81}{x}.$$

i. Find the average rate of change of P on the interval $[1, 9]$.

At $x = 1$, $P = 15 + 12 - 81 = -54$. At $x = 9$, $P = 15 + 36 - 9 = 42$. The average rate of change is:

$$\frac{42 - (-54)}{9 - 1} = \frac{96}{8} = 12.$$

ii. Find a formula for the marginal profit. What is the marginal profit for $x = 9$?

Marginal profit is just P' :

$$P' = 0 + 12\left(\frac{1}{2\sqrt{x}}\right) - 81((-1)x^{-2}) = \frac{6}{\sqrt{x}} + \frac{81}{x^2}.$$

$$\text{At } x = 9, P'(9) = \frac{6}{3} + \frac{81}{81} = 2 + 1 = 3.$$

(b) A sphere of radius 3 has its radius changing at a rate of 4 inches per second. How quickly is the volume of the sphere changing?

We have: $r = 3$, $\frac{dr}{dt} = 4$.

We want: $\frac{dV}{dt}$

We know: $V = \frac{4}{3}\pi r^3$

By related rates, we have

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt},$$

so at $r = 3$,

$$\frac{dV}{dt} = 4\pi(3)^2(4) = 144\pi.$$

4. Rules of Differentiation (40 points)

(a) For each of the following functions, find the derivative. (30 points; 10 points each)

i. $f(x) = \frac{2\sqrt{x}+1}{\sqrt{x}}$

$$\begin{aligned} f'(x) &= \frac{(\sqrt{x})(2(\frac{1}{2\sqrt{x}})) - (2\sqrt{x} + 1)(\frac{1}{2\sqrt{x}})}{\sqrt{x}^2} \\ &= \frac{1 - 1 - \frac{1}{2\sqrt{x}}}{x} \\ &= \frac{-1}{2x^{3/2}} \end{aligned}$$

ii. $g(t) = t^3 + 3t^2 - 5$

$$g'(t) = 3t^2 + 6t.$$

iii. $h(x) = \frac{2x+1}{x+2}$

$$h'(x) = \frac{(x+2)(2) - (2x+1)(1)}{(x+2)^2} = \frac{3}{(x+2)^2}.$$

(b) At which values for x does the graph of $y = -x^3 + 9x$ have a horizontal tangent line? (10 points)

The slope of the tangent line to the graph of $y = -x^3 + 9x$ is given by the derivative,

$$y' = -3x^2 + 9.$$

A horizontal line has slope 0. If $y' = 0$ then $x = \pm\sqrt{3}$.

5. **Rates of Change** (30 points)

(a) The profit P from selling x units of a product is given by

$$P = 30 + 12\sqrt{x} - \frac{81}{x}.$$

Find a formula for the marginal profit. What is the marginal profit for $x = 9$? (10 points)

Marginal profit is just P' :

$$P' = 0 + 12\left(\frac{1}{2\sqrt{x}}\right) - 81((-1)x^{-2}) = \frac{6}{\sqrt{x}} + \frac{81}{x^2}.$$

$$\text{At } x = 9, P'(9) = \frac{6}{3} + \frac{81}{81} = 2 + 1 = 3.$$

(b) Consider $f(x) = (x^2 - x + 2)(x^2 + x)$.

i. What is the average rate of change of $f(x)$ on the interval $[0, 1]$? (10 points)

$$\frac{f(1) - f(0)}{1 - 0} = (1 - 1 + 2)(1 + 1) - (0 - 0 + 2)(0 + 0) = 4.$$

ii. What is the instantaneous rate of change of $f(x)$ at $x = 0$, and at $x = 1$? (10 points)

The instantaneous rate of change is the derivative:

$$f'(x) = (x^2 + x)(2x - 1) + (x^2 - x + 2)(2x + 1) = 4x^3 + 2.$$

$$\text{At } x = 0, f'(x) = 2, \text{ and at } x = 1, f'(x) = 4 + 2 = 6.$$

6. **True/False** (30 points, 5 points each)

Mark each question as (**T**)rue or (**F**)alse.

- a. T The derivative of \sqrt{x} is always non-negative.
- b. T π radians = 180 degrees.
- c. T A derivative of a position function with respect to time is a velocity.
- d. F To apply the Quotient Rule to $f(x)/g(x)$, we must have that $g(x)$ is always positive.
- e. F $\cos(\pi - \theta) = \cos \theta$.
- f. F If a function is continuous at a point, then it is differentiable at that point.

7. Derivative Tests

- (a) Apply the Second-Derivative Test to find the relative extrema of $f(x) = x^3 - 3x^2 + 1$. Show all of your work.

$$f'(x) = 3x^2 - 6x$$

$$f''(x) = 6x - 6$$

Critical numbers of f :

$$f'(x) = 0 \implies x = 0, 2$$

f' undefined nowhere.

By the Second-Derivative Test, $f''(0) = -6 \implies$ relative maximum at $x = 0$;

$f''(2) = 6 \implies$ relative minimum at $x = 2$.

- (b) Apply the First-Derivative Test to find the relative extrema of $g(t) = t^4 - 4t^3 - 4$. Show all of your work.

$$g'(t) = 4t^3 - 12t^2$$

Critical numbers of g :

$$g'(t) = 0 \implies t = 0, 3$$

g' undefined nowhere.

Intervals	$(-\infty, 0)$	$(0, 3)$	$(3, \infty)$
Sign of g'	$g'(-1) = -$	$g'(1) = -$	$g'(4) = +$
Conclusion	increasing	increasing	decreasing

$t = 0$ is *not* a relative extremum, but $t = 3$ is a relative maximum.

8. Optimization

- (a) Find two positive numbers such that their product is 75, and so that the first plus three times the second is a minimum.

We have: $xy = 75$

We want: minimize $S = x + 3y$

We know: nothing more.

$xy = 75 \implies y = \frac{75}{x}$, so $S = x + \frac{225}{x}$. Thus,

$$S' = 1 - \frac{225}{x^2} = \frac{x^2 - 225}{x^2}.$$

Our feasible domain is $x > 0$. The critical numbers of S are:

$$S' = 0 \implies x = \pm 15;$$

$$S' \text{ undefined} \implies x = 0.$$

The only feasible critical number is $x = 15$, which is a minimum:

$$S'' = \frac{450}{x^3} \implies S''(15) = +.$$

The answer is: $x = 15$, $y = 5$.

- (b) A rectangular page is to contain 18 square inches of print. The margins at the top and bottom are .5 inches and on each side are 1 inch. What dimensions minimize the amount of paper used?

Let $A := \text{Printable Area}$, and let $B := \text{Page Area}$.

We have: $A = 18$

We want: minimize B

We know: $A = (x-2)(y-1)$, $B = xy$

$A = (x-2)(y-1) = 18 \implies y = \frac{18}{x-2} + 1$, so $B = x(\frac{18}{x-2} + 1) = \frac{18x}{x-2} + x$. Thus,

$$B' = \frac{(x-2)(18) - (18x)(1)}{(x-2)^2} + 1 = \frac{18x - 36 - 18x + x^2 - 4x + 4}{(x-2)^2} = \frac{(x-8)(x+4)}{(x-2)^2}.$$

Our feasible domain is $x > 2$.

The critical numbers of B are:

$$B' = 0 \implies x = 8, -4;$$

$$B' \text{ undefined} \implies x = 2.$$

The only feasible critical number is $x = 8$, which is a minimum:

Intervals	$(2, 8)$	$(8, \infty)$
Sign of B'	$B'(4) = \frac{-+}{+} = -$	$B'(10) = \frac{++}{+} = +$
Conclusion	decreasing	increasing

When $x = 8$, $y = \frac{18}{6} + 1 = 4$. The answer is: 8×4 .

9. Sketching Graphs (60 points)

Sketch the graphs of the following functions, showing all work: state domain and range; label all asymptotes, points of discontinuity, and intercepts; and show how you obtained relative extrema, points of inflection.

(a) $g(x) = \frac{x^2}{x^2-1}$

$$\begin{aligned}g(x) &= \frac{x^2}{(x+1)(x-1)} \\g'(x) &= \frac{(x^2-1)(2x) - (x^2)(2x)}{(x^2-1)^2} = \frac{-2x}{(x^2-1)^2} \\g''(x) &= \frac{(x^2-1)^2(-2) - (-2x)2(x^2-1)(2x)}{(x^2-1)^4} \\&= \frac{(x^2-1)(-2x^2+2+8x^2)}{(x^2-1)^4} = \frac{(x^2-1)(6x^2+2)}{(x^2-1)^4}\end{aligned}$$

Domain: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

Horizontal Asymptote at $g(x) = \frac{1}{1} = 1$

Vertical Asymptotes at $x = \pm 1$

Critical numbers of g :

$$g'(x) = 0 \implies x = 0;$$

$$g' \text{ undefined} \implies x = \pm 1$$

Critical numbers $x = \pm 1$ are from asymptotes; by Second-Derivative Test ($g''(0) = -2$), $x = 0$ is a relative maximum.

Critical numbers of g' :

$$g''(x) = 0 \text{ nowhere;}$$

$$g'' \text{ undefined} \implies x = \pm 1.$$

Since there are asymptotes at $x = \pm 1$, there are no points of inflection.

Range: $(-\infty, 0] \cup (1, \infty)$

(see graphing calculator for sketch; for instance, www.coolmath.com/graphit)

6. Sketching Graphs (continued)

(b) $f(x) = \frac{x^2}{x^2+1}$

$$\begin{aligned} f(x) &= 1 - (x^2 + 1)^{-1} \\ f'(x) &= 2x(x^2 + 1)^{-2} \\ f''(x) &= 2(x^2 + 1)^{-2} - 8x^2(x^2 + 1)^{-3} = 2\frac{1 - 7x^2}{(x^2 + 1)^3}. \end{aligned}$$

Domain: $(-\infty, \infty)$

Horizontal Asymptote at $f(x) = \frac{1}{1} = 1$
 No Vertical Asymptotes (no zeros of denominator)

Critical numbers of f :

$$f'(x) = 0 \implies x = 0;$$

f' undefined nowhere

By Second-Derivative Test ($f''(0) = 2$), $x = 0$ is a relative minimum.

Critical numbers of f' :

$$f''(x) = 0 \implies x = \pm\frac{1}{\sqrt{7}};$$

f'' undefined nowhere.

Testing concavity, we see both $x = \pm\frac{1}{\sqrt{7}}$ are points of inflection:

Intervals	$(-\infty, -\frac{1}{\sqrt{7}})$	$(-\frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}})$	$(\frac{1}{\sqrt{7}}, \infty)$
Sign of f''	$f''(-1) = \frac{-}{+} = -$	$f''(0) = \frac{+}{+} = +$	$f''(1) = \frac{-}{+} = -$
Conclusion for f :	concave down	concave up	concave down

Range: $[0, 1)$

(see graphing calculator for sketch; for instance, www.coolmath.com/graphit)

10. True/False

Mark each question as (**T**)rue or (**F**)alse.

- a. F The absolute maximum of a function on a closed interval never occurs at the endpoints.
- b. F The derivative of $\sec(x)$ is $-\sec(x)\tan(x)$.
- c. T A function can only change from increasing to decreasing at a critical number.
- d. F A function cannot intersect its horizontal asymptotes.
- e. T If $f'(x) > 0$ for all x , then f is concave upward for all x .
- f. T The Product Rule states: $\frac{d}{dx}[f(x)g(x)] = g(x)f'(x) + f(x)g'(x)$.
- g. T A point of inflection of f can occur *only* at a critical number of f' .

11. Chain Rule (40 points)

- (a) The number of bacteria in a culture after t days is modeled by

$$N(t) = 400 \left(1 - \frac{4}{(t^2 + 3)^2} \right).$$

How many bacteria are there when $t = 1$? What is the rate of change of the bacteria population when $t = 1$? (15 points)

At $t = 1$, $N(1) = 400(1 - \frac{4}{4^2}) = 400\frac{3}{4} = 300$.

The rate of change of $N(t)$ in general is:

$$N'(t) = 400((-4)(-2)(t^2 + 3)^{-3}(2t)) = 6400t(t^2 + 3)^{-3}.$$

At $t = 1$,

$$N'(1) = 6400(1)(4)^{-3} = 100.$$

- (b) Find the equation of the tangent line to the graph of $f(x) = (9 - x^3)^{\frac{-1}{2}}$ when $x = 2$. (15 points)

We have:

$$f'(x) = \left(\frac{-1}{2}\right)(9 - x^3)^{\frac{-3}{2}}(-3x^2) = \frac{3x^2}{2(9 - x^3)^{\frac{3}{2}}}.$$

When $x = 2$, $f(2) = 1$ and $f'(2) = 6$. Thus, the tangent line is:

$$(y - 1) = 6(x - 2) \quad \implies \quad y = 6x - 11.$$

- (c) Compute the derivative of $f(t) = \sqrt{t^2 + 4} - \sqrt{t^2 - 4}$. (10 points)

Rewriting, we have $f(t) = (t^2 + 4)^{\frac{1}{2}} - (t^2 - 4)^{\frac{1}{2}}$. Thus,

$$\begin{aligned} f'(t) &= \frac{1}{2}(t^2 + 4)^{\frac{-1}{2}}(2t) - \frac{1}{2}(t^2 - 4)^{\frac{-1}{2}}(2t) \\ &= \frac{t}{\sqrt{t^2 + 4}} - \frac{t}{\sqrt{t^2 - 4}} \\ &= t \frac{\sqrt{t^2 - 4} - \sqrt{t^2 + 4}}{\sqrt{t^2 + 4}\sqrt{t^2 - 4}}. \end{aligned}$$

12. Trig Functions (40 points)

(a) Show that, if $y = \frac{5 - \cos x}{x}$, then $xy' + y = \sin x$. (10 points)

We have that

$$y' = \frac{x \sin x - (5 - \cos x)(1)}{x^2} = \frac{\sin x}{x} - \frac{5 - \cos x}{x^2}.$$

Thus,

$$xy' + y = x \frac{\sin x}{x} - x \frac{5 - \cos x}{x^2} + \frac{5 - \cos x}{x} = \sin x.$$

(b) Find the derivatives of the following functions. (30 points; 10 points each)

i. $y = x \tan(5x)$

$$y' = (\tan(5x))(1) + x(\sec^2(5x))(5) = \tan(5x) + 5x \sec^2(5x) \quad \left(= \frac{5x + \sin(5x) \cos(5x)}{\cos^2(5x)} \right).$$

ii. $T = x - 4 \csc(\pi x + 1)$

$$T' = 1 - 4(-\csc(\pi x + 1) \cot(\pi x + 1))(\pi) = 1 + 4\pi \csc(\pi x + 1) \cot(\pi x + 1).$$

iii. $f(t) = \sqrt{\sin^2 t - \cos^2 t}$

Since $f(t) = (\sin^2 t - \cos^2 t)^{\frac{1}{2}}$, we have:

$$\begin{aligned} f'(t) &= \frac{1}{2}(\sin^2 t - \cos^2 t)^{-\frac{1}{2}} (2(\sin t)(\cos t) - 2(\cos t)(-\sin t)) \\ &= \frac{2(\sin t)(\cos t)}{\sqrt{\sin^2 t - \cos^2 t}}. \end{aligned}$$

13. **Higher-Order Derivatives and Implicit Differentiation** (50 points)

(a) A ball is propelled straight upward from ground level on Earth with an initial velocity of $80ft/s$.

i. What is the height of the ball after t seconds? (5 points)

The height is given by $s(t) = -16t^2 + 80t$.

ii. When is the ball at its highest point? (15 points)

The ball is at its highest point when $s' = 0$. Since

$$s'(t) = -32t + 80,$$

we have $s'(t) = 0$ when $t = 2.5$.

(b) Find $\frac{dy}{dx}$ if $\sec(xy) = 5$. (15 points)

Using implicit differentiation, we have:

$$\begin{aligned}\frac{d}{dx}(\sec(xy)) &= \frac{d}{dx}(5) \implies \\ \sec(xy) \tan(xy) \frac{d}{dx}(xy) &= 0 \implies \\ \sec(xy) \tan(xy) \left(y + x \frac{dy}{dx}\right) &= 0 \implies \\ \frac{dy}{dx} &= -\frac{y}{x}.\end{aligned}$$

Also acceptable:

$$\frac{dy}{dx} = -\frac{\operatorname{arcsec}(5)}{x^2}.$$

- (c) What is the rate of change of t with respect to y (*not* y with respect to t !) if $y = \sqrt{t+1}$? (15 points)

Using implicit differentiation, we have:

$$\begin{aligned}\frac{d}{dy}(y) &= \frac{d}{dy}((t+1)^{\frac{1}{2}}) \implies \\ 1 &= \frac{1}{2}(t+1)^{-\frac{1}{2}} \frac{d}{dy}(t+1) \implies \\ 2\sqrt{t+1} &= \frac{dt}{dy}.\end{aligned}$$

Also acceptable:

$$\frac{dt}{dy} = 2y.$$

14. Related Rates (40 points)

- (a) You are given that the instantaneous rate of change for x with respect to time is 5 when $x = 3$. If $y = 1 + 2xy$, what is the corresponding value of $\frac{dy}{dt}$? (15 points)

We have: $\frac{dx}{dt} = 5$, $x = 3$.

We want: $\frac{dy}{dt}$.

We know: $y = 1 + 2xy$.

$$\begin{aligned}y &= 1 + 2xy \implies \\ \frac{dy}{dt} &= 2 \frac{d}{dt}(xy) \implies \\ \frac{dy}{dt} &= 2x \frac{dy}{dt} + 2y \frac{dx}{dt} \implies \\ \frac{dy}{dt} &= \frac{2y \frac{dx}{dt}}{1 - 2x}.\end{aligned}$$

When $x = 3$, $y = \frac{-1}{5}$, so

$$\frac{dy}{dt} = \frac{\frac{-2}{5}(5)}{-5} = \frac{2}{5}.$$

- (b) The radius r of a sphere is increasing by 3 feet per minute.
- What is the rate of change of the volume V when $r = 10$ feet? (15 points)

We have: $\frac{dr}{dt} = 3$, $r = 10$.

We want: $\frac{dV}{dt}$.

We know: $V = \frac{4\pi r^3}{3}$.

$$\begin{aligned}V &= \frac{4\pi}{3} r^3 \implies \\ \frac{dV}{dt} &= \frac{4\pi}{3} (3r^2 \frac{dr}{dt}) = 4\pi r^2 \frac{dr}{dt}.\end{aligned}$$

When $r = 10$,

$$\frac{dV}{dt} = 4\pi(10)^2(3) = 1200\pi.$$

ii. What is $\frac{dV}{dt}$ when $V = 36\pi$? (10 points)

When $V = 36\pi$, since $V = \frac{4\pi r^3}{3}$, we have that $r = 3$. Thus, by Part i,

$$\frac{dV}{dt} = 4\pi(3)^2(3) = 108\pi.$$

15. **True/False** (30 points, 5 points each)

Mark each question as (**T**)rue or (**F**)alse.

- a. F The surface area of a sphere of radius r is $\frac{4}{3}\pi r^2$.
- b. F If $f(x)$ is an n^{th} degree polynomial, then $f^{(n)}(x) = 0$.
- c. T The General Power rule is an example of the Chain Rule, applied to the Simple Power Rule.
- d. F Acceleration is the second derivative of velocity.
- e. F We used the Chain Rule to prove that $\frac{d}{dt}[\sin(t)] = \cos(t)$.
- f. F $\csc(2t) = \frac{2}{\sin(t)}$.