Math 16A Short Calculus, section 2

Practice Midterm, Part 1

SOLUTIONS

1. **True/False**

   Mark each question as (**T**)rue or (**F**)alse.

   a. _T_ The line $(y-3) = 2(x+4)$ is in point-slope form.

   b. _F_ If $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$, then $f \circ g(x) = \frac{1}{x^3}$.

   c. _T_ If $p(x)$ is a polynomial function and c is any real number, then $\lim_{x\to c} p(x) = p(c)$.

   d. _F_ Two lines are parallel if and only if their slopes are negative reciprocals of each other.

   e. _F_ All continuous functions are differentiable.

   f. _F_ The graph of $y = \frac{1}{x^2+5}$ has a vertical asymptote.

   g. _F_ Infinity is a real number.

   h. _F_ The equation $x^2 + 2y^2 = 48$ represents a circle.

   i. _T_ The graph of $y = 2x^2 + 2$ is symmetrical in the y-axis.

   j. _F_ If an equation passes the horizontal line test, then the equation defines a function.

2. **Lines and Circles**

(a) Consider the points $(2, -3)$ and $(-2, 3)$.

i. What is the distance between the points?

$$\begin{aligned} d((2,-3),(-2,3)) &= \sqrt{(2-(-2))^2+(-3-3))^2} \\ &= \sqrt{4^2+6^2} = \sqrt{52} = 2\sqrt{13}. \end{aligned}$$

ii. What is their midpoint?
The midpoint is at $\left(\frac{2+(-2)}{2}, \frac{-3+3}{2}\right) = (0, 0)$.

iii. Write the line containing them in slope-intercept form, and in point-slope form for the point $(2, -3)$.
The slope of the line is $m = \frac{-3-3}{2-(-2)} = \frac{-6}{4} = -\frac{3}{2}$.
The y-intercept is 0.
So in point-slope form the equation of the line is $(y + 3) = -\frac{3}{2}(x - 2)$.
In slope-intercept form it is $y = -\frac{3}{2}x$.

(b) Consider the circle whose general form is

$$3x^2 + 3y^2 - 6x + 18y - 18 = 0.$$

i. Write the equation in standard form.
We have $x^2 + y^2 - 2x + 6y - 6 = 0$.
So $x^2 - 2x + y^2 + 6y - 6 = 0$.
So $(x - 1)^2 - 1 + (y + 3)^2 - 9 - 6 = 0$.
So $(x - 1)^2 + (y + 3)^2 = 16$.
So $(x - 1)^2 + (y + 3)^2 = 4^2$.

ii. State the center, radius, and x-intercepts.
The center is $(1, -3)$ and the radius is 4.
For the x-intercepts, set $y = 0$.
So $(x - 1)^2 = 16 - 9 = 7$.
So $x = 1 \pm \sqrt{7}$.
So the x-intercepts are $1 + \sqrt{7}$ and $1 - \sqrt{7}$.

3. **Functions**

Let $f(x) = x^2 - 2$ and $g(x) = \sqrt{x-1}$.

(a) Sketch the graphs of each function, labelling intercepts.

(b) State an appropriate domain for $g(x)$. What is the range of $g(x)$?

A good domain is $[1, \infty)$. The range is $[0, \infty)$.

(c) Compute the following:

i. $f(x) - g(x)$

$$f(x) - g(x) = x^2 - 2 - \sqrt{x-1}.$$

ii. $\frac{f(x)}{g(x)}$

$$\frac{f(x)}{g(x)} = \frac{x^2 - 2}{\sqrt{x-1}}.$$

iii. $g \circ f(x)$

$$g \circ f(x) = \sqrt{x^2 - 3}.$$

(d) Find $g^{-1}(x)$. What is its domain?

Put $y = \sqrt{x-1}$.
So $y^2 = x - 1$.
So $x = y^2 + 1$.
Hence $g^{-1}(x) = x^2 + 1$. Its domain is the range of g, $[0, \infty)$.

4. **Limits**

Evaluate, showing your working, the following limits.

(a) $\lim_{x \to 3} 3x + 3$

$f(x) = 3x + 3$ is continuous.

So,

$$\lim_{x \to 3} 3x + 3 = 3 \cdot 3 + 3 = 12.$$

(b) $\lim_{x \to -1} \frac{x^2 - 1}{x + 1}$

$$\lim_{x \to -1} \frac{x^2 - 1}{x + 1} = \lim_{x \to -1} \frac{(x + 1)(x - 1)}{x + 1} = \lim_{x \to -1} \frac{(x - 1)}{1} = -1 - 1 = -2.$$

(c) $\lim_{x \to 3^+} \frac{|x - 3|}{x - 3}$

$$\lim_{x \to 3^+} \frac{|x - 3|}{x - 3} = \lim_{x \to 3^+} \begin{cases} 1 & x > 3 \\ -1 & x < 3 \end{cases} = 1.$$

5. **Continuity**

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let a be a real number. What does it mean to say that f is continuous at a?

$f : \mathbb{R} \to \mathbb{R}$ is continuous at a if $\lim_{x \to a} f(x)$ exists and equals $f(a)$.

(b) Give an example of a function for which $\lim_{x \to 3} f(x)$ exists but the function is not continuous at 3. Carefully sketch the graph of your function, taking care to show what happens near 3.

6. **Differentiation**

(a) Find the equation of the tangent to $y = x^3$ at $(3, 9)$. Write your final answer in slope intercept form.

We have

$$\frac{dy}{dx} = 3x^2.$$

So when $x = 3$, $\frac{dy}{dx} = 3 \times 3^2 = 18$.

Hence the equation of the tangent is $y - 9 = 18(x - 3)$.

Thus $y = 18x - 54 + 9$.

Hence $y = 18x - 45$.