Unknotting crossing changes and circular Heegaard splittings

Alexander Coward

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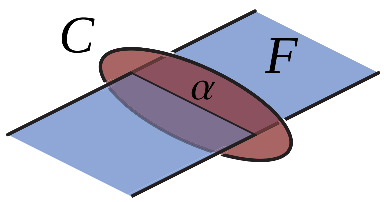
Question: Is there an algorithm to determine whether a given knot has unknotting number one?

Let F be a compact surface with boundary, embedded in a 3-manifold M. Let D be a disc embedded in M so that $D \cap F$ is a single arc α properly embedded in F and embedded in the interior of D. Let $C = \partial D$.



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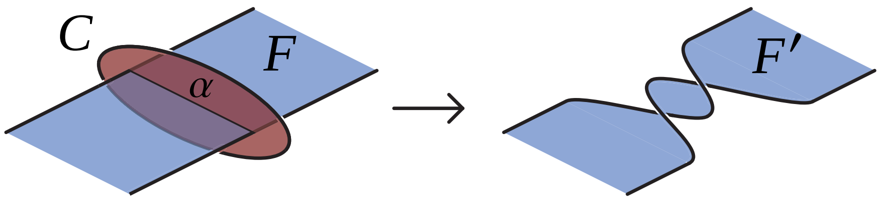

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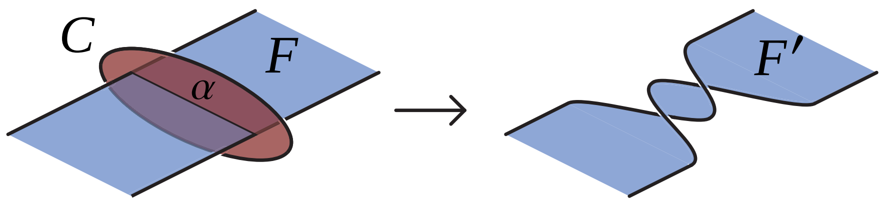

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Observation: If F and F' are related by a twist, then the links ∂F and $\partial F'$ are related by a crossing change.

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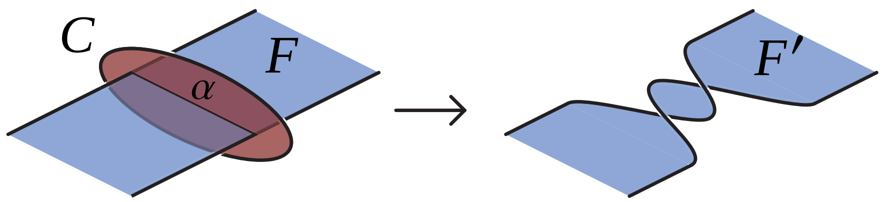

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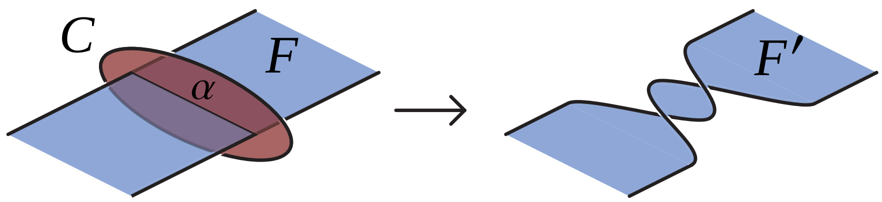

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Question: Is there an algorithm to determine whether two surfaces are related by a single twist, up to ambient isotopy?

Corollary 1: [C] If the answer to this question is 'yes', then there is an algorithm to determine whether a hyperbolic knot in S^3 has unknotting number one.

Theorem 1: [C] Let K and K' be oriented knots in S^3 where $g(K) > g(K')$ and K and K' are both either hyperbolic or fibered. Then, up to ambient isotopy fixing K (resp. K'), there are finite lists of oriented spanning surfaces $\{S_1, \ldots, S_n\}$ for K and $\{S'_1, \ldots, S'_{n'}\}$ for K' with the property that if K and K' are related by a single crossing change, then some S_i and some S'_j are related by a single twist.

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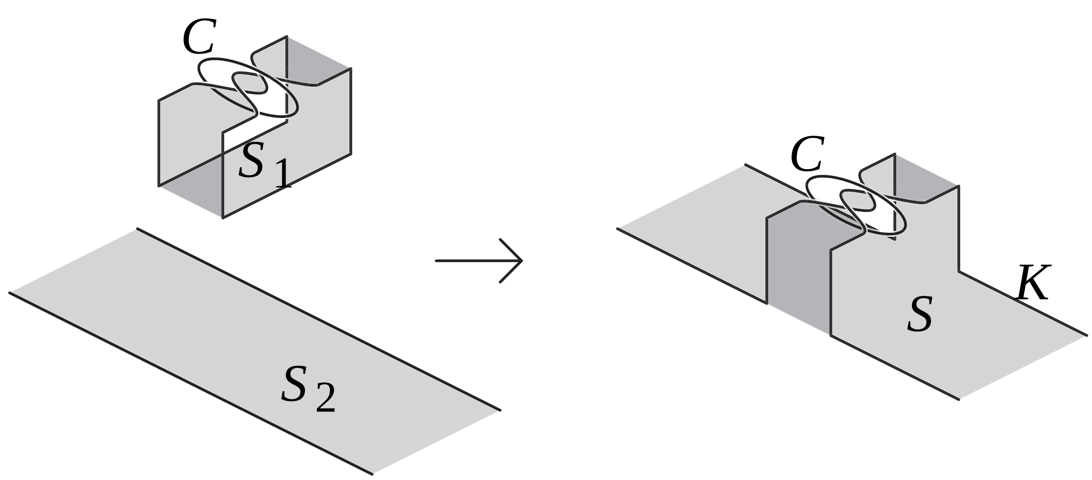
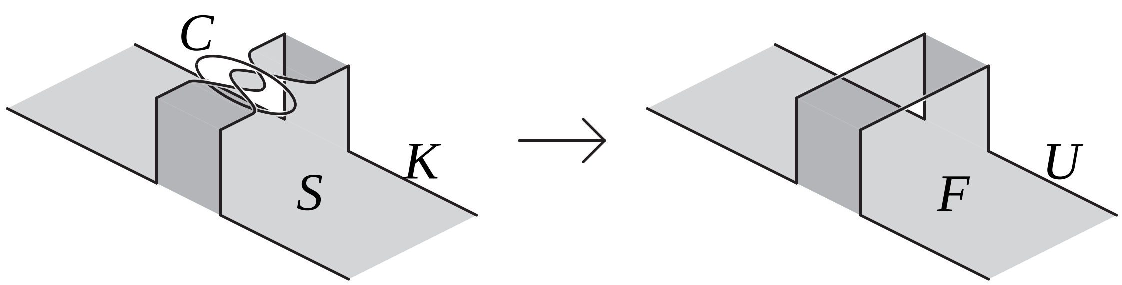
Furthermore, there is an algorithm that will take diagrams for K and K' as input, and output such finite lists of spanning surfaces.

Corollary 2: [C] If there is an algorithm to determine whether two compact, oriented surfaces in S^3, each with a single boundary component, are related by a single twist, up to ambient isotopy, then there is an algorithm to determine whether two knots satisfying the hypotheses of Theorem 1 are related by a single crossing change.

Theorem: [Scharlemann and Thompson, 1989] Let C be a crossing circle for a non-trivial knot K such that performing a crossing change along C unknots K. Then K has a minimal genus Seifert surface, S, which is obtained by plumbing two surfaces S_1 and S_2, where S_1 is a Hopf band. Moreover, there is an ambient isotopy, keeping K fixed throughout, that takes C to the associated crossing circle for S_1.



The unknotting crossing change from the Scharlemann–Thompson theorem untwists the Hopf band.



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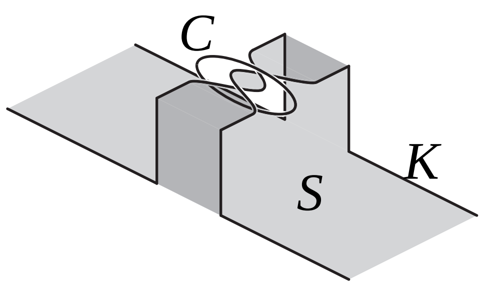

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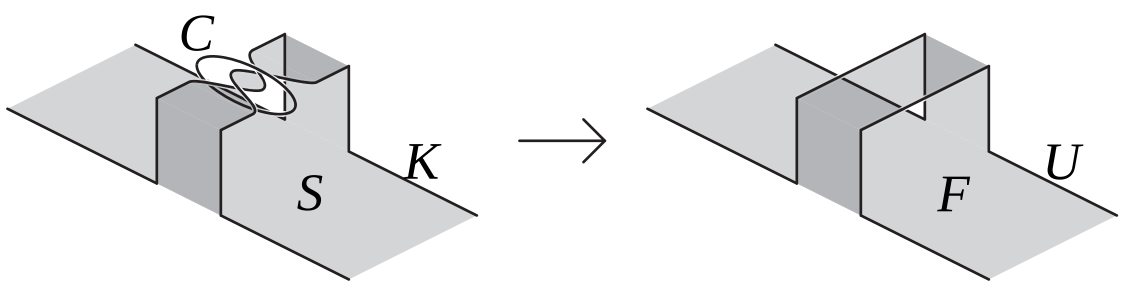

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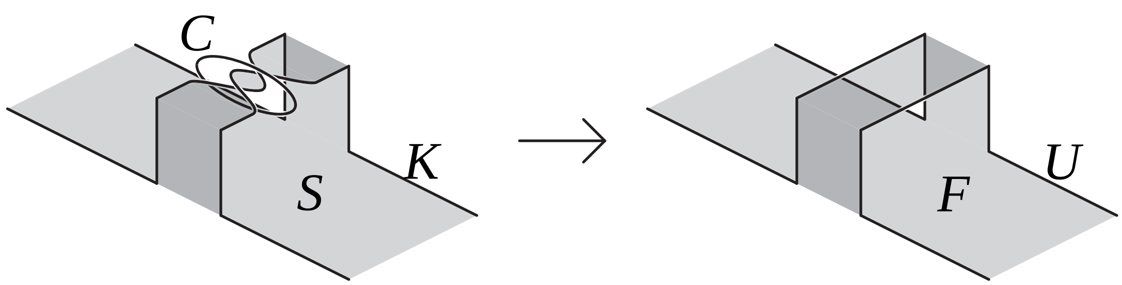

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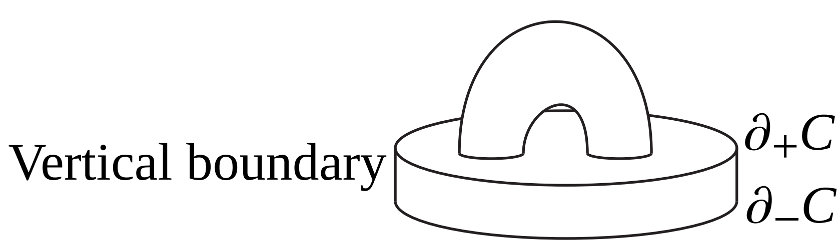
We shall use ideas for the theory of Heegaard splittings.

Circular Heegaard Splittings

A compression body, C, is a compact connected 3-manifold homeomorphic to the result of adjoining 1-handles to $S \times \{1\}$ in $S \times I$ and capping off 2-sphere boundary components of $S \times \{0\}$, where S is a compact orientable surface.

We call $\partial S \times I$ the vertical boundary. The closure of the rest of ∂C is the horizontal boundary .

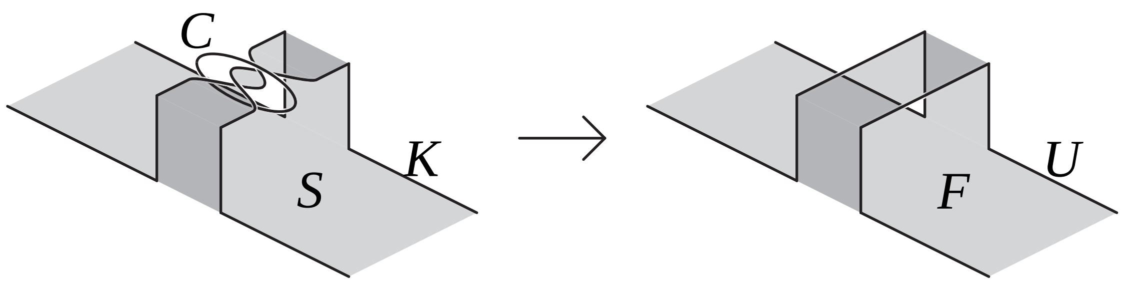
We denote $S \times \{0\}$ minus capped off 2-sphere components by $\partial_- C$ and the closure of $\partial C - (\partial_+ C \cup (\partial S \times I))$ by $\partial_+ C$.



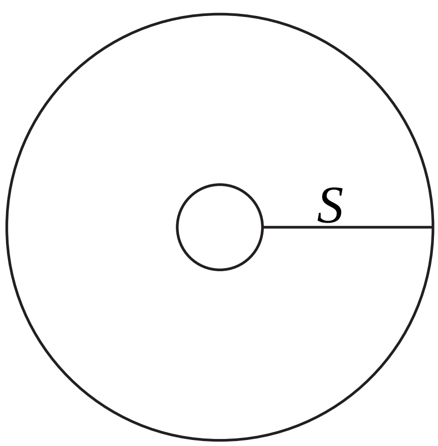
A decomposition of a compact orientable 3-manifold M along a surface S into two compression bodies, C_1 and C_2, where $S = \partial_+ C_1 = \partial_+ C_2$, is called a Heegaard splitting. We call S the corresponding Heegaard surface.

A decomposition of a compact orientable 3-manifold M along two surfaces S_+ and S_- into two compression bodies, C_1 and C_2, where $S_+ = \partial_+ C_1 = \partial_+ C_2$ and $S_- = \partial_- C_1 = \partial_- C_2$, is called a circular Heegaard splitting. We call $S_+ \cup S_-$ the corresponding circular Heegaard surface. We call S_+ (resp. S_-) the positive (resp. negetive) surface of the splitting. We call $g(S_-)$ the genus of the splitting, and $g(S_+) - g(S_-)$ the handle number of the splitting.

What can we say about F?



Suppose K is a fibered knot.



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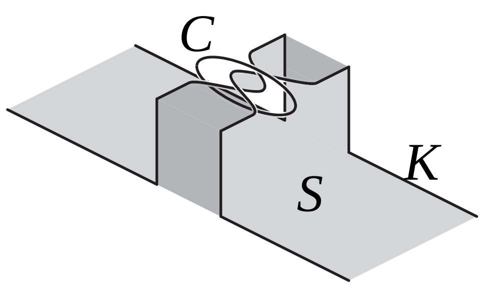

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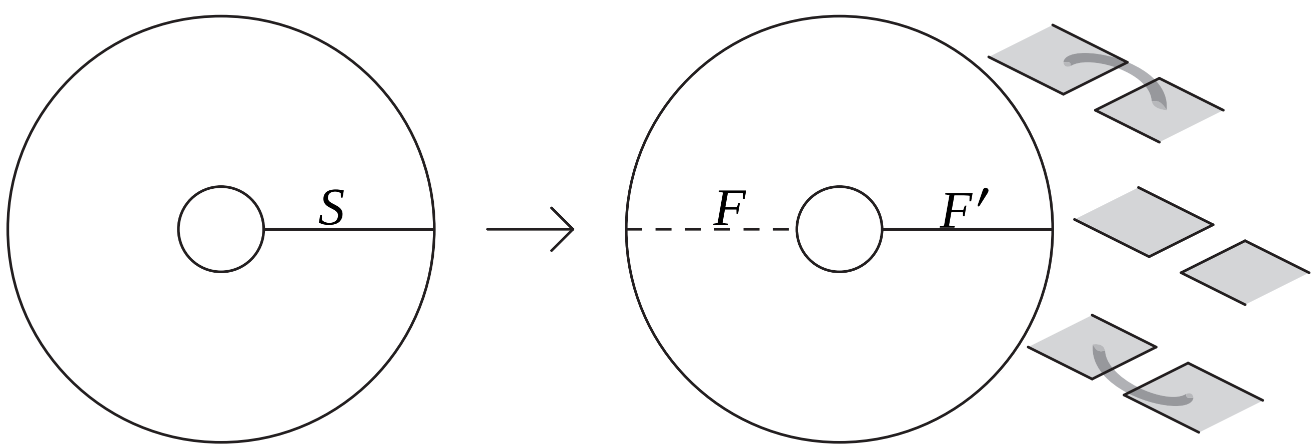

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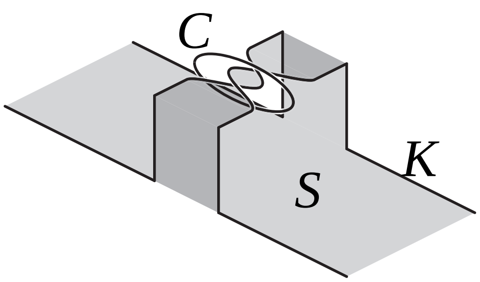

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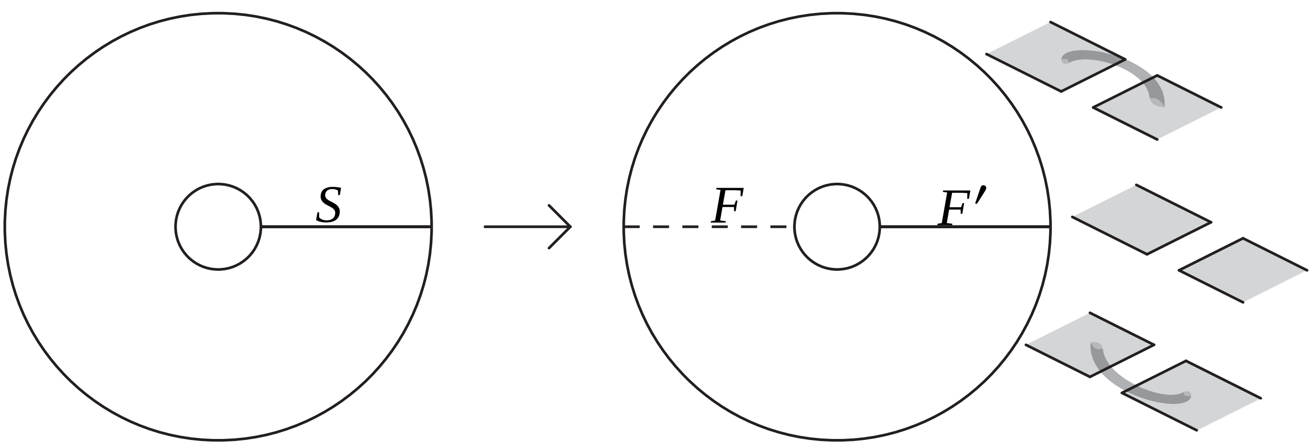
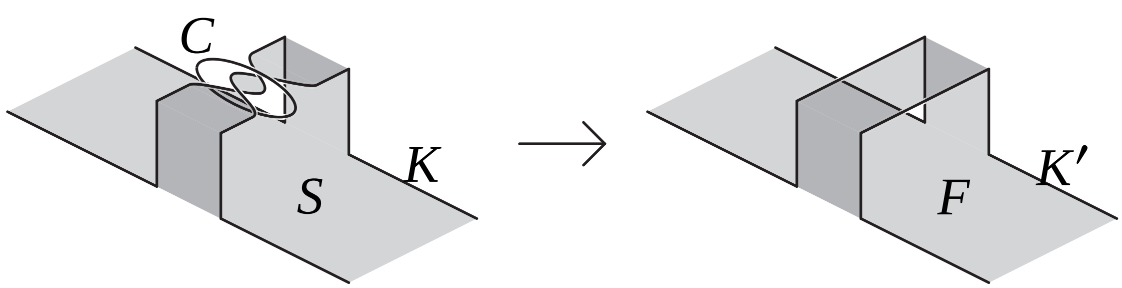


We obtain a handle number one circular Heegaard splitting of the complement of U, with positive surface F.

Theorem: [Essentially Kobayashi] Let K' be a genus n fibered knot. For any integer $m \geq n$ the exterior of K' has a unique handle number one, genus m circular Heegaard splitting, up to ambient isotopy. Furthermore there is an algorithm to find it, starting with any diagram for K' and the integer m.

See “Fibered links and unknotting operations”



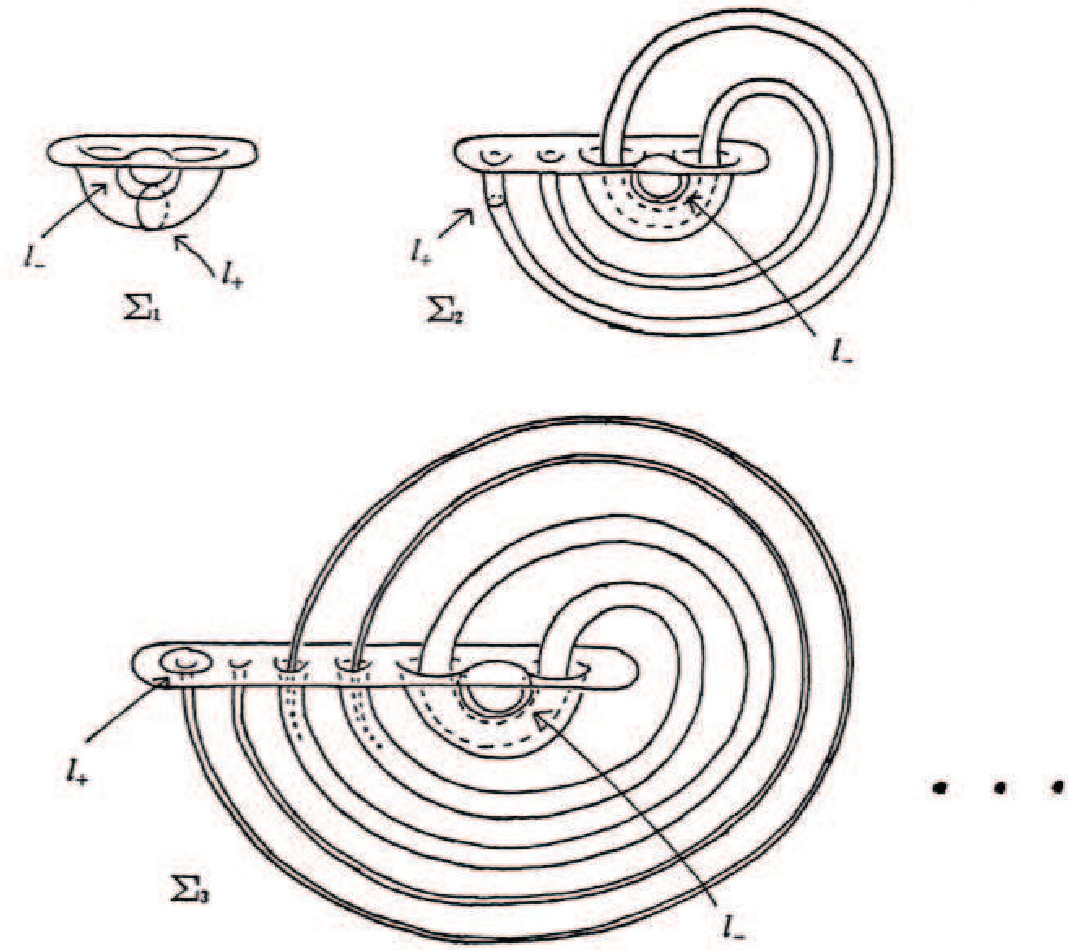


Fig. 1.3

Theorem: [Lackenby] Let M be a complete finite-volume hyperbolic 3-manifold with non-empty boundary. Then there is an algorithm to determine the Heegaard genus of M. Moreover, for any given positive integer n, there is an algorithm to find all Heegaard surfaces for M with genus at most n, up to ambient isotopy.

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Idea of Proof:

Use (almost) normal surface theory to search for generalized Heegaard splittings with strongly irreducible thick surfaces and incompressible thin surfaces.

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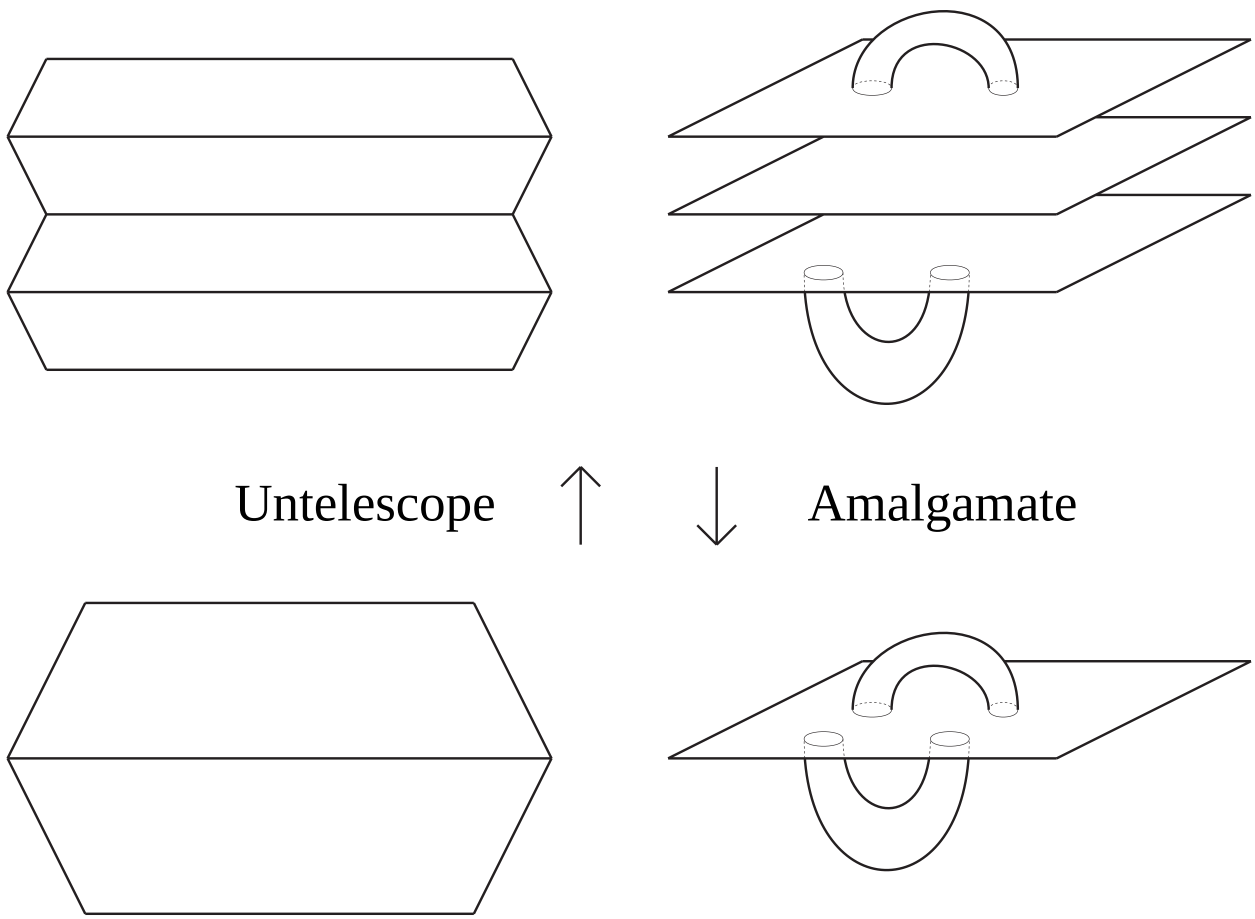
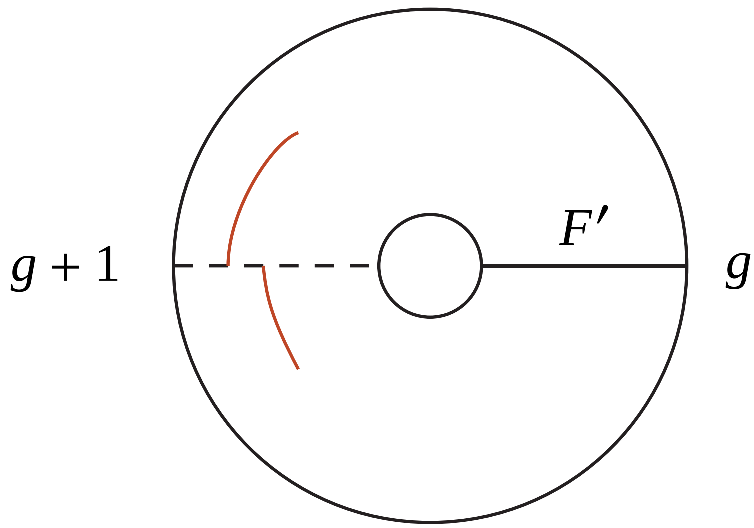
Key Proposition: [Lackenby] If one amalgamates a generalised Heegaard splitting, the resulting Heegaard splitting is well-defined up to ambient isotopy. In particular, it is independent of the order of partial amalgamations and the choice of handle structures on the compression bodies.


Untelescope
Amalgamate

Consider a handle number one circular Heegaard splitting of the exterior of the unknot.



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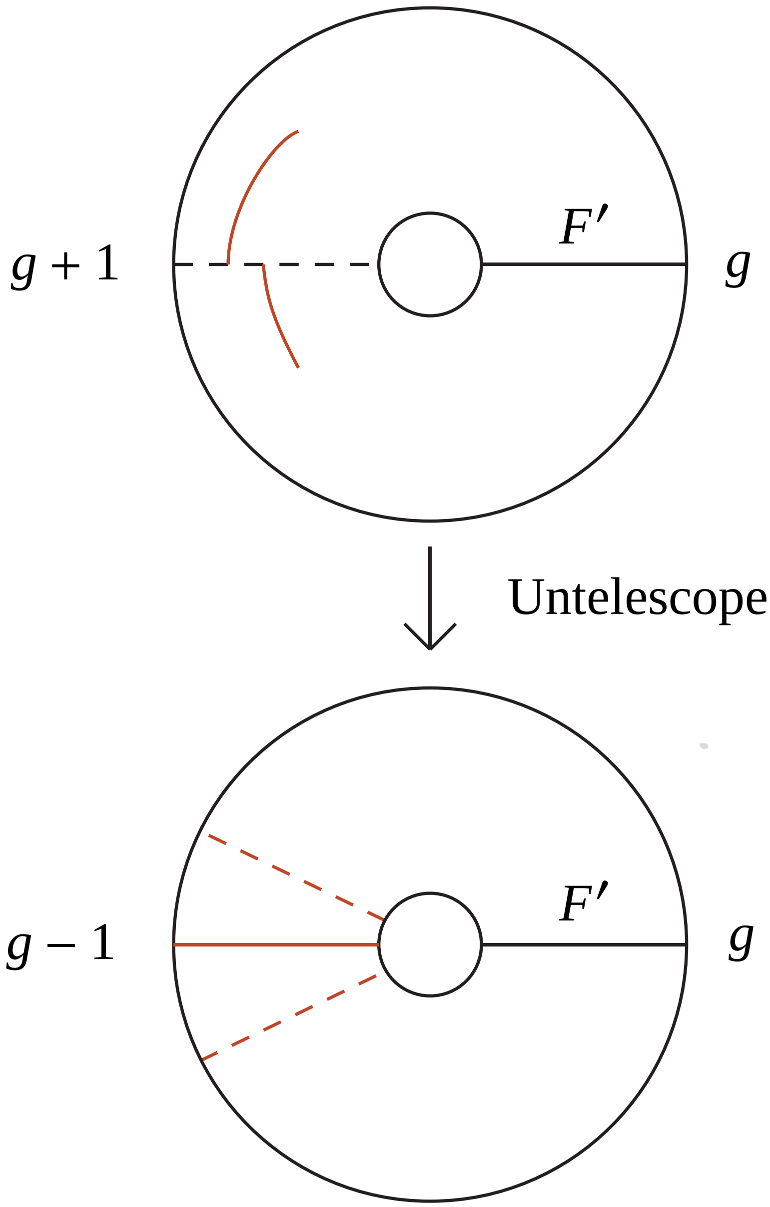

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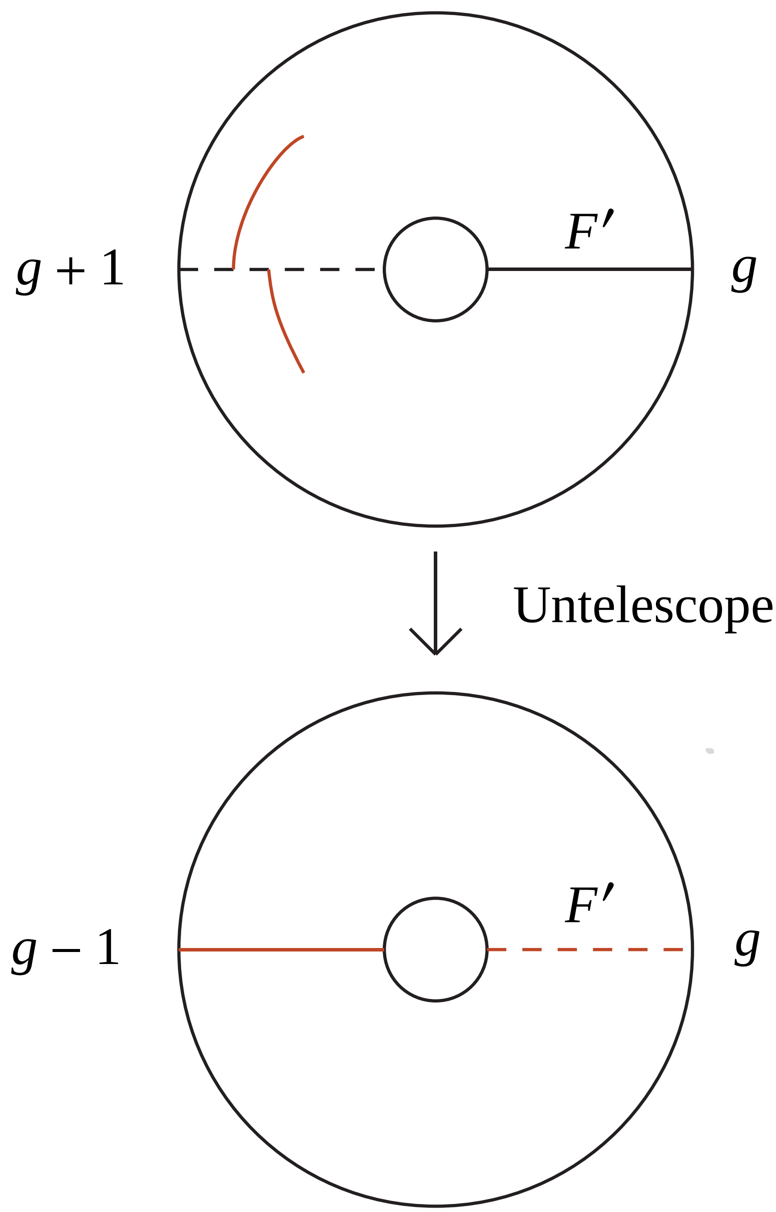

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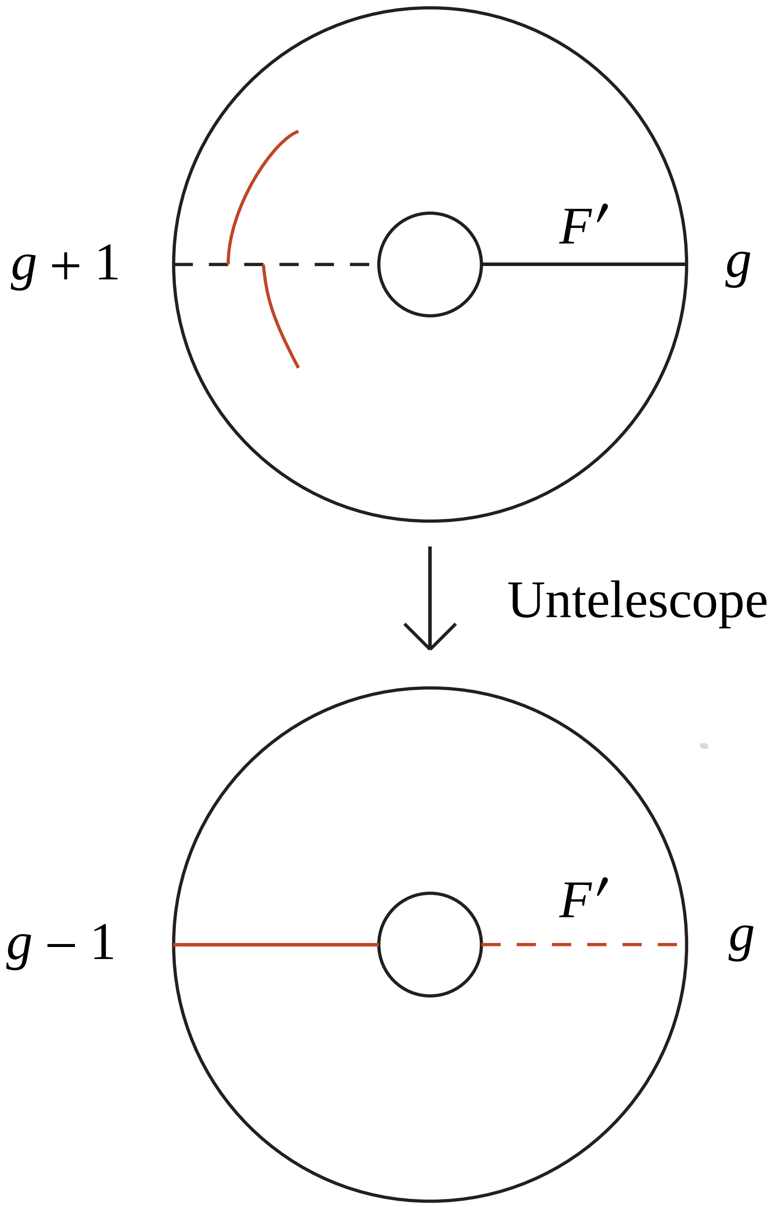

Repeat.

Eventually the negative surface must be incompressible. But, the only incompressible spanning surface for the unknot is a disc. Further, it follows from a theorem of Scharlemann and Thompson that the positive surface must then be a stabilized disc.

So to recapture the handle number one, genus m, circular Heegaard splitting for the unknot, amalgamate this until the desired genus is reached.

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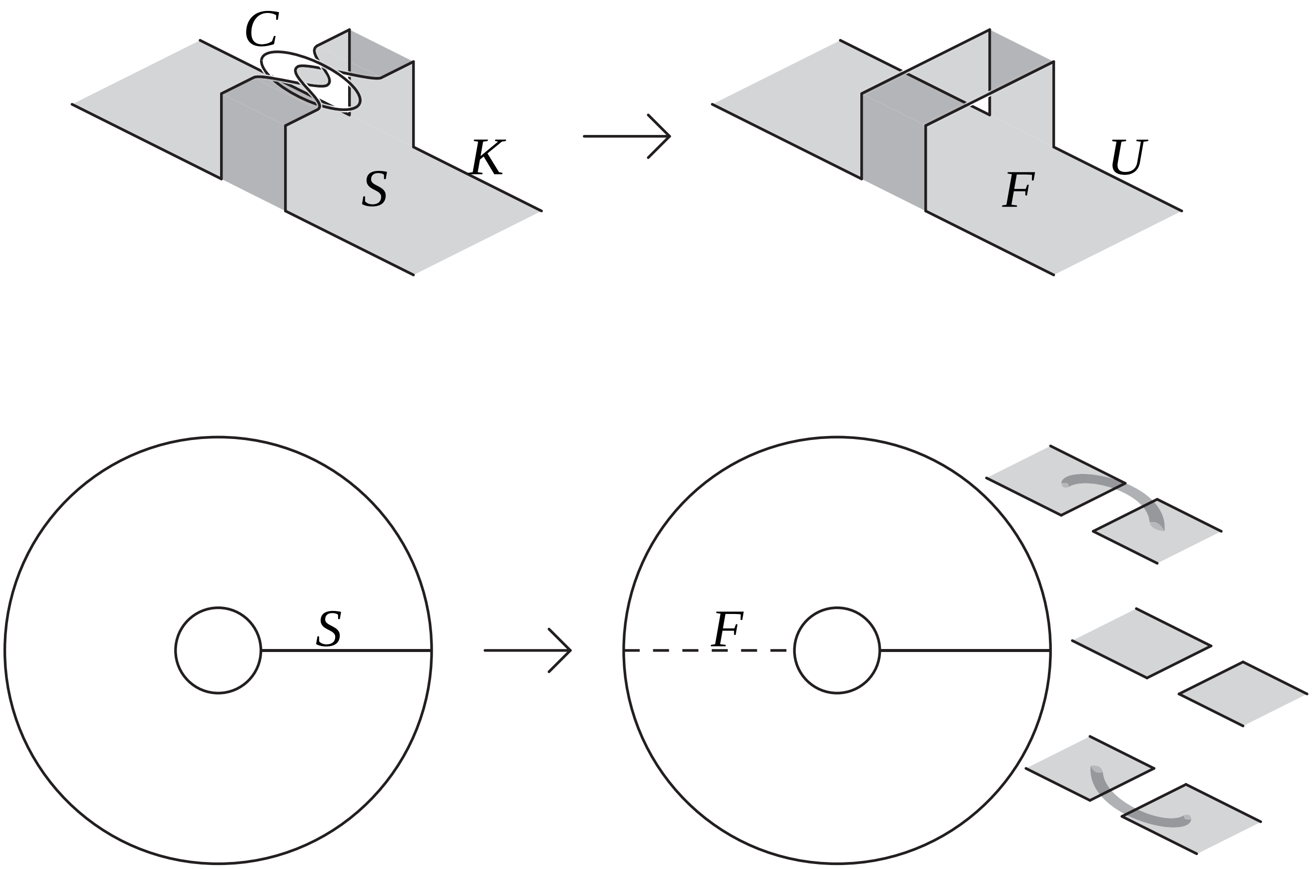
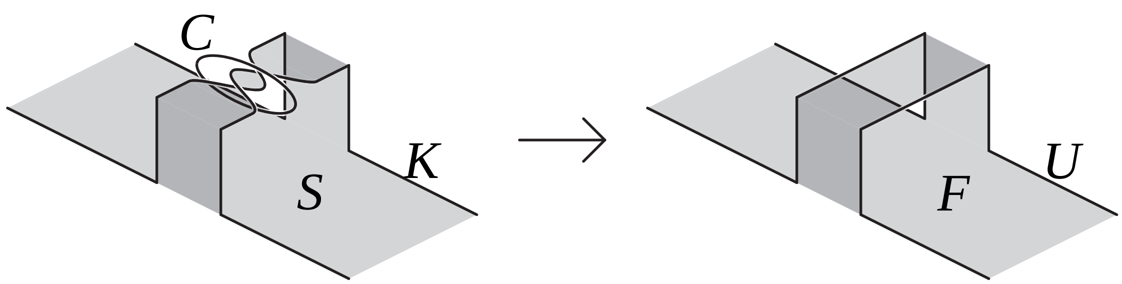
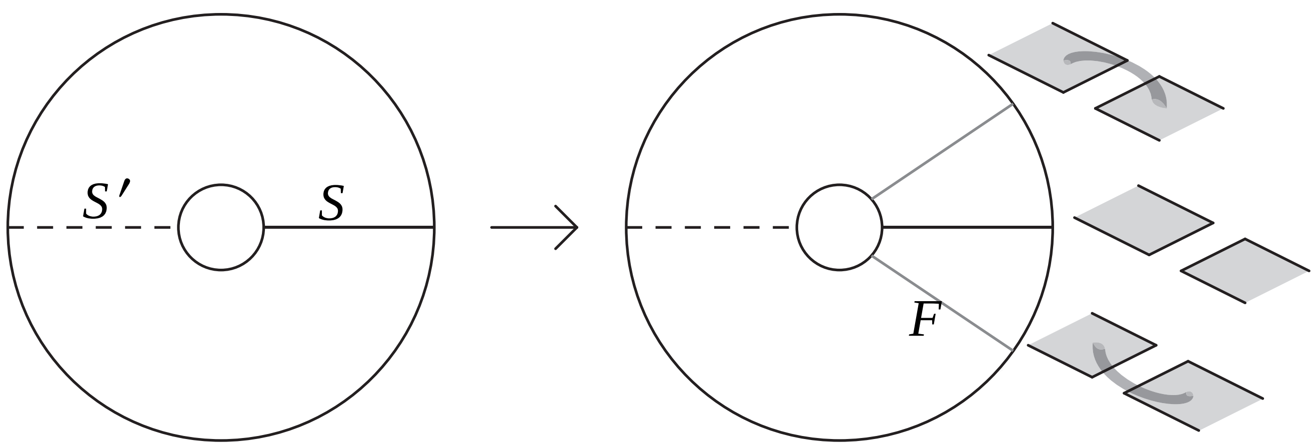
What if K is not fibered?

Before (when *K* is fibered) we had this:



Now we have this:





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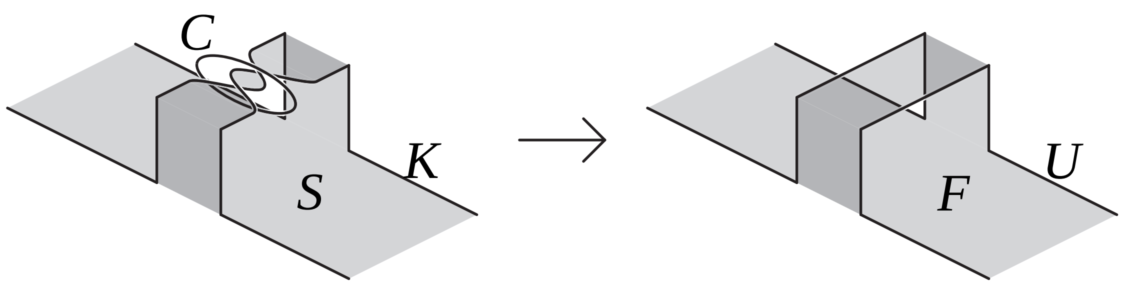

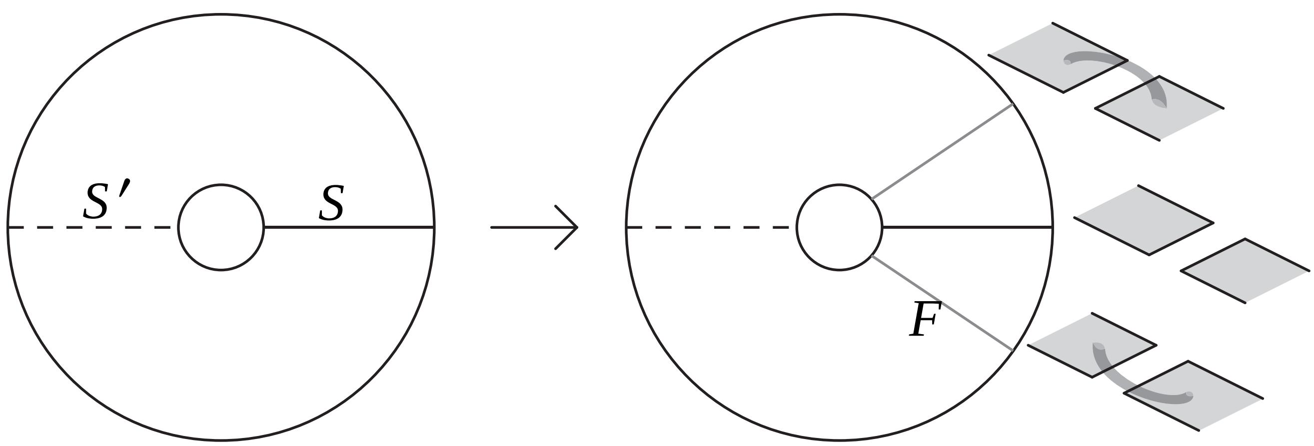

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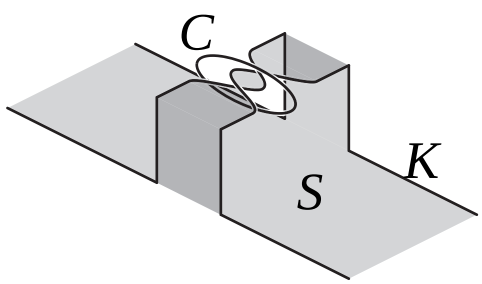

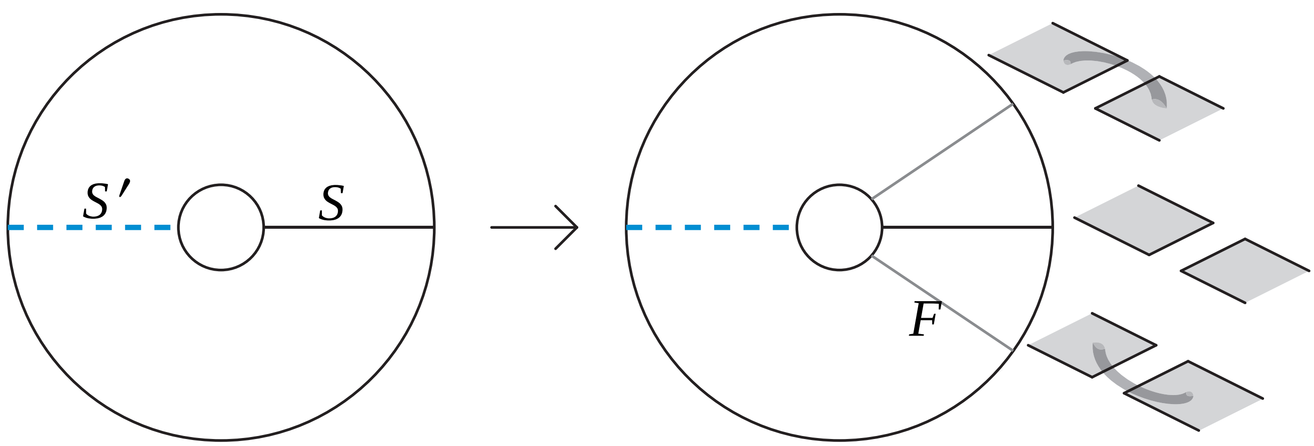

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Solution: Look at the positive surfaces.

Theorem 2: [C] Let K be either a hyperbolic knot or a fibered knot. Then the exterior of K has at most finitely many circular Heegaard surfaces with given genus and handle number, up to ambient isotopy. Furthermore, given these numbers and a diagram for K, there is an algorithm to find them all such surfaces.

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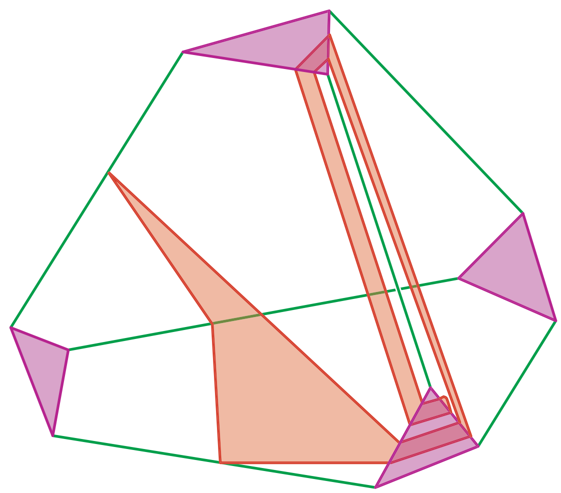

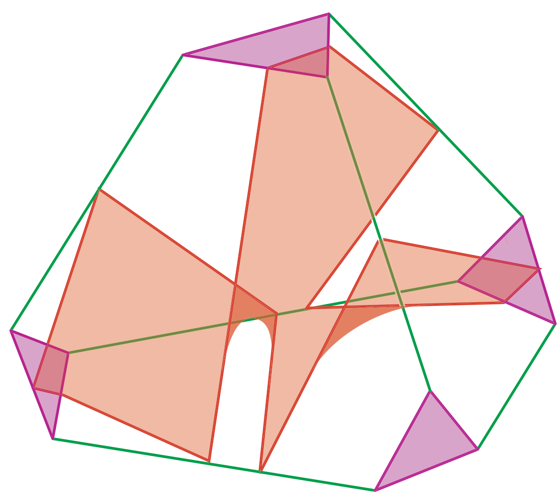

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THE END