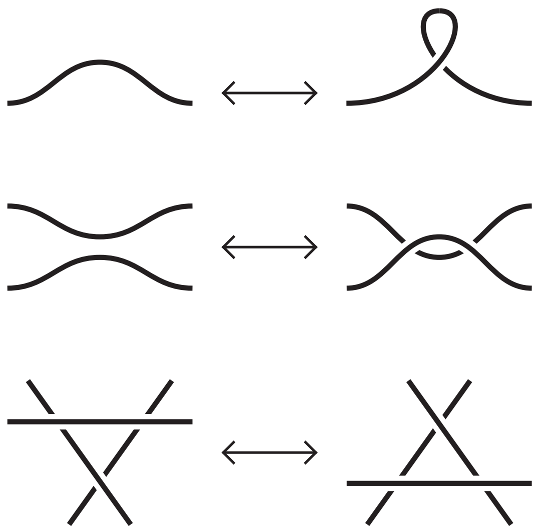
An upper bound for Riedemeister moves

Alexander Coward
University of California
at Davis

Joint work with Marc Lackenby

Theorem: [Reidemeister 1926, Alexander and Briggs 1927]
Any two diagrams of the same knot or link may be joined by a finite sequence of Reidemeister moves.



Question: How many moves are needed for a given number of crossings in each diagram?

Question: How many moves are needed for a given number of crossings in each diagram?

More precisely: Is there a computable function $F : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all diagrams D_1 and D_2 of a link, L, with n_1 and n_2 crossings respectively, one may turn D_1 into D_2 with at most $F(n_1, n_2)$ Reidemeister moves?

Question: How many moves are needed for a given number of crossings in each diagram?

More precisely: Is there a computable function $F : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all diagrams D_1 and D_2 of a link, L, with n_1 and n_2 crossings respectively, one may turn D_1 into D_2 with at most $F(n_1, n_2)$ Reidemeister moves?

Answer:

Question: How many moves are needed for a given number of crossings in each diagram?

More precisely: Is there a computable function $F : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all diagrams D_1 and D_2 of a link, L, with n_1 and n_2 crossings respectively, one may turn D_1 into D_2 with at most $F(n_1, n_2)$ Reidemeister moves?

Answer: Yes.

Question: How many moves are needed for a given number of crossings in each diagram?

More precisely: Is there a computable function $F : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all diagrams D_1 and D_2 of a link, L, with n_1 and n_2 crossings respectively, one may turn D_1 into D_2 with at most $F(n_1, n_2)$ Reidemeister moves?

Answer: Yes.

Equivalent to the recognition problem for links, solved by Haken and Hemion.

Question: How many moves are needed for a given number of crossings in each diagram?

More precisely: Is there a computable function $F : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all diagrams D_1 and D_2 of a link, L, with n_1 and n_2 crossings respectively, one may turn D_1 into D_2 with at most $F(n_1, n_2)$ Reidemeister moves?

Answer: Yes.

Equivalent to the recognition problem for links, solved by Haken and Hemion.

This F is computable but not explicit.

Theorem: [Hass and Lagarias, 1998] If L is the unknot then $2^{cn_1} + 2^{cn_2}$ Reidemeister moves suffice, where $c = 10^{11}$.

Theorem: [Hass and Lagarias, 1998] If L is the unknot then $2^{cn_1} + 2^{cn_2}$ Reidemeister moves suffice, where $c = 10^{11}$.

Hayashi (2004): (Very large) bounds for splitting links.

Theorem: [Hass and Lagarias, 1998] If L is the unknot then $2^{cn_1} + 2^{cn_2}$ Reidemeister moves suffice, where $c = 10^{11}$.

Hayashi (2004): (Very large) bounds for splitting links.

Theorem: [C, Lackenby] If L is any knot or link then $F(n_1, n_2) = \exp^{(c^n)}(n)$ moves suffice where $c = 2^{400}$.

Here $\exp(n) = 2^n$.

Overview of Hass-Lagarias:

Overview of Hass-Lagarias:

Let D be an unknot diagram with n crossings.

Overview of Hass-Lagarias:

Let D be an unknot diagram with n crossings.

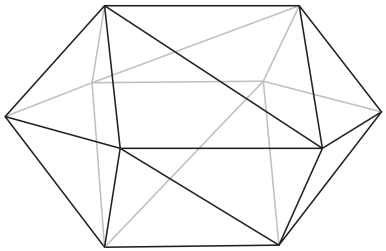
We want to turn D into the trivial unknot diagram with no crossings.

Overview of Hass-Lagarias:

Let D be an unknot diagram with n crossings.

We want to turn D into the trivial unknot diagram with no crossings.

Build B, a compact convex triangulated polyhedron in $\mathbb{R}^3$.

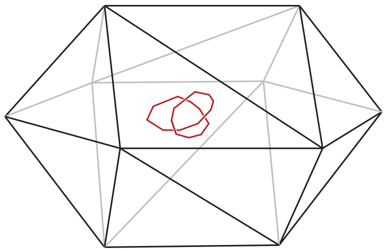


Overview of Hass-Lagarias:

Let D be an unknot diagram with n crossings.

We want to turn D into the trivial unknot diagram with no crossings.

Build B, a compact convex triangulated polyhedron in $\mathbb{R}^3$.



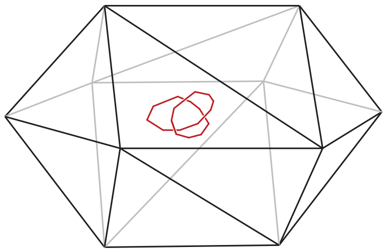
Contains a knot, K, in its 1-skeleton which projects to D under $(x, y, z) \to (x, y)$.

Overview of Hass-Lagarias:

Let D be an unknot diagram with n crossings.

We want to turn D into the trivial unknot diagram with no crossings.

Build B, a compact convex triangulated polyhedron in $\mathbb{R}^3$.



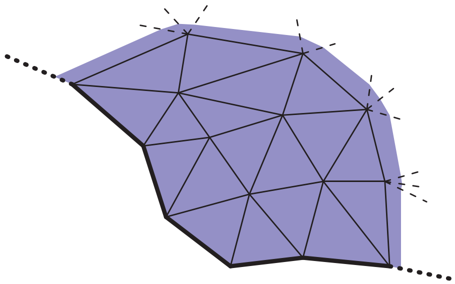
Contains a knot, K, in its 1-skeleton which projects to D under $(x, y, z) \to (x, y)$.

Number of straight tetrahedra $< 840n$.

Observation: K bounds a disc, S.

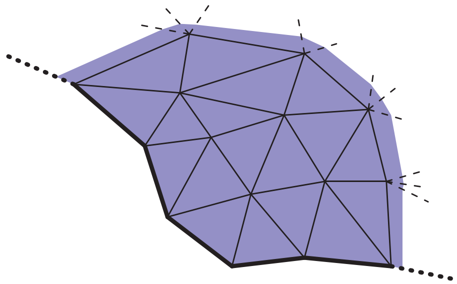
Observation: K bounds a disc, S.

Idea: Arrange S to consist of a bounded number of flat triangles.

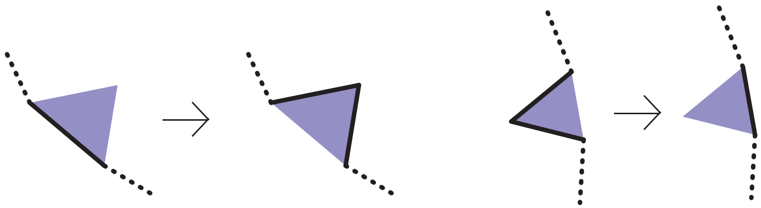


Observation: K bounds a disc, S.

Idea: Arrange S to consist of a bounded number of flat triangles.

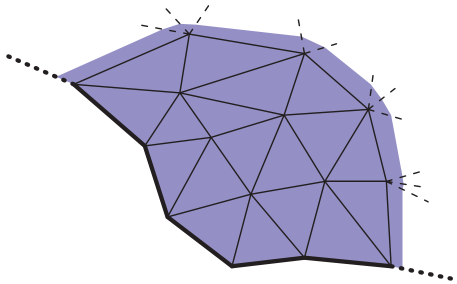


Then slide across S with elementary moves.

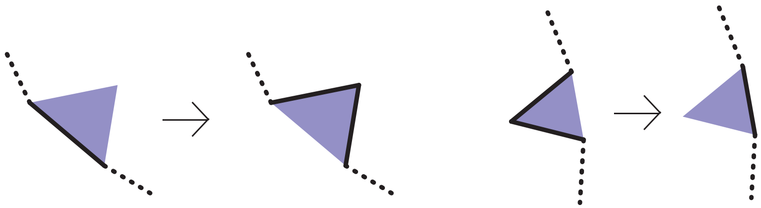


Observation: K bounds a disc, S.

Idea: Arrange S to consist of a bounded number of flat triangles.



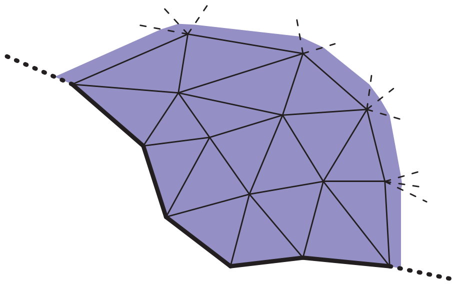
Then slide across S with elementary moves.



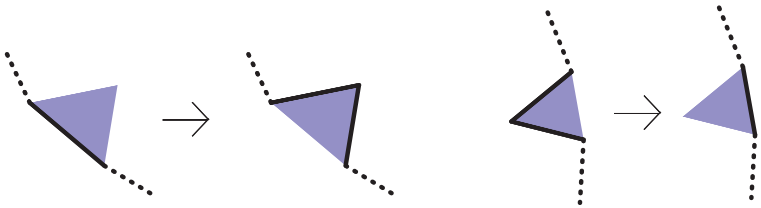
Normal surface theory provides us with such a bound.

Observation: K bounds a disc, S.

Idea: Arrange S to consist of a bounded number of flat triangles.



Then slide across S with elementary moves.



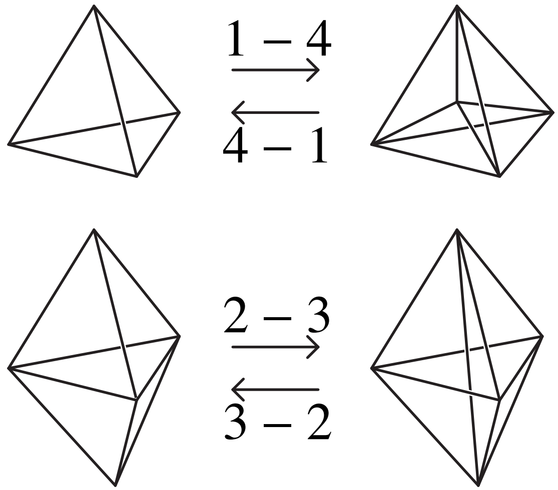
Normal surface theory provides us with such a bound.

The resulting elementary moves project to at most $2^{10^{11}n}$ Reidemeister moves.

Pachner moves

Let T_1 and T_2 be triangulations for M, a 3-manifold.

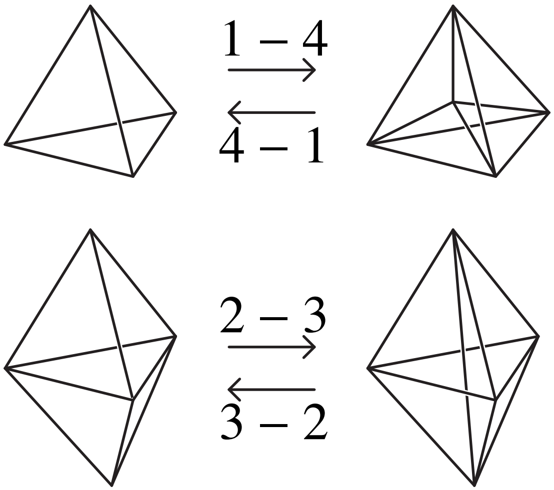
Theorem: [Pachner] T_1 and T_2 may be interchanged via Pachner moves.



Pachner moves

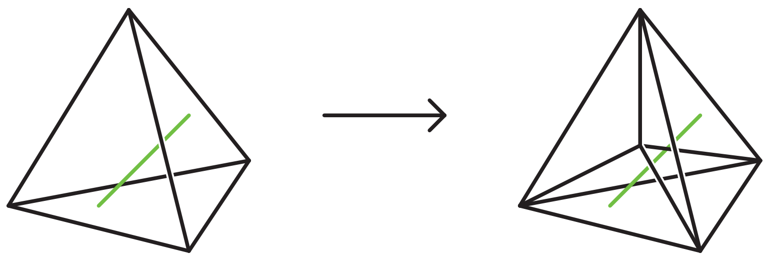
Let T_1 and T_2 be triangulations for M, a 3-manifold.

Theorem: [Pachner] T_1 and T_2 may be interchanged via Pachner moves.

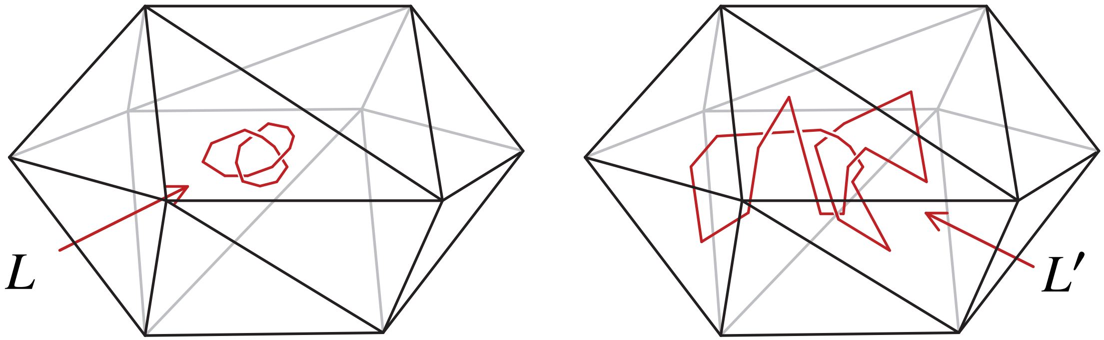


Mijatoviċ: Bound on the number of Pachner moves required for a large class of 3-manifolds, including all link exteriors.

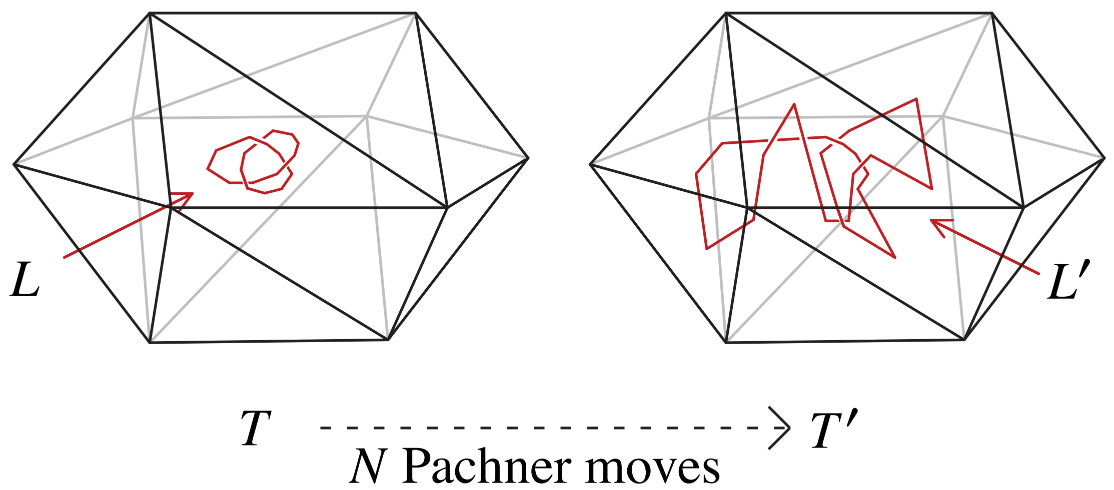
Observation: A Pachner move $T \rightsquigarrow T'$ induces a PL-homeomorphism $|T| \to |T'|$ which sends every straight arc to a concatenation of at most 4 straight arcs.



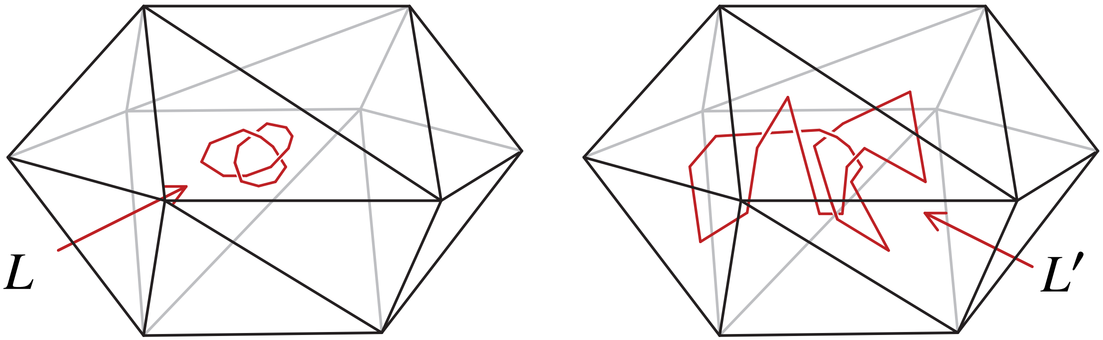
Setup for the main theorem:



Setup for the main theorem:



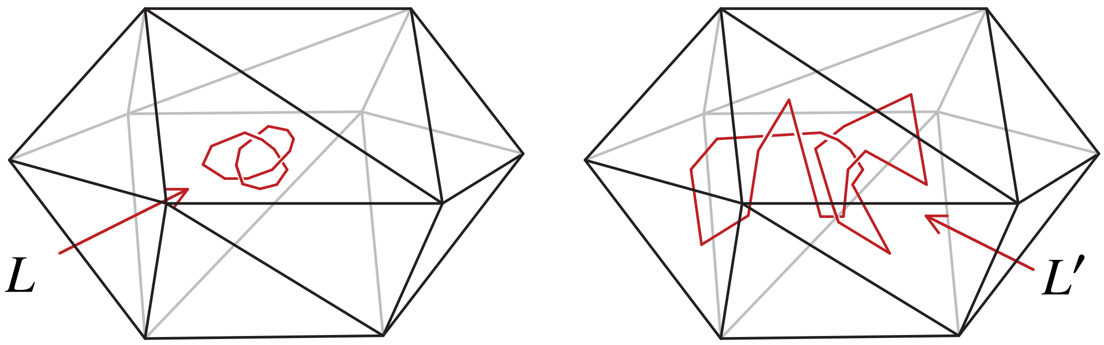
Setup for the main theorem:



$$T \dashrightarrow T' \quad N \text{ Pachner moves}$$

$$|T| \xrightarrow{h} |T'|$$

Setup for the main theorem:

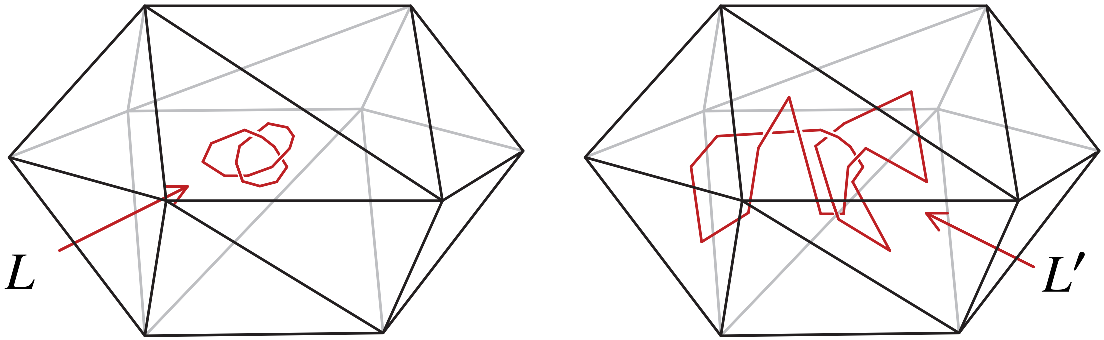


$$T \overset{N \text{ Pachner moves}}{\dashrightarrow} T'$$

$$|T| \xrightarrow{\;h\;} |T'|$$

$$L \longmapsto L'$$

Setup for the main theorem:



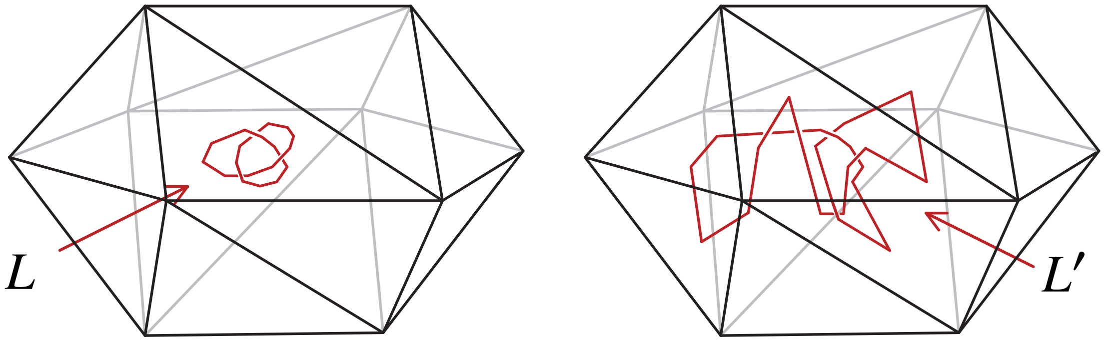
T - - - - - - → T' (N Pachner moves)

$|T| \xrightarrow{\;h\;} |T'|$

$L \longmapsto L'$

Straight arc $\longmapsto$ Concatenation of $\leq 4^N$ straight arcs

Setup for the main theorem:



$T \dashrightarrow T'$ (N Pachner moves)

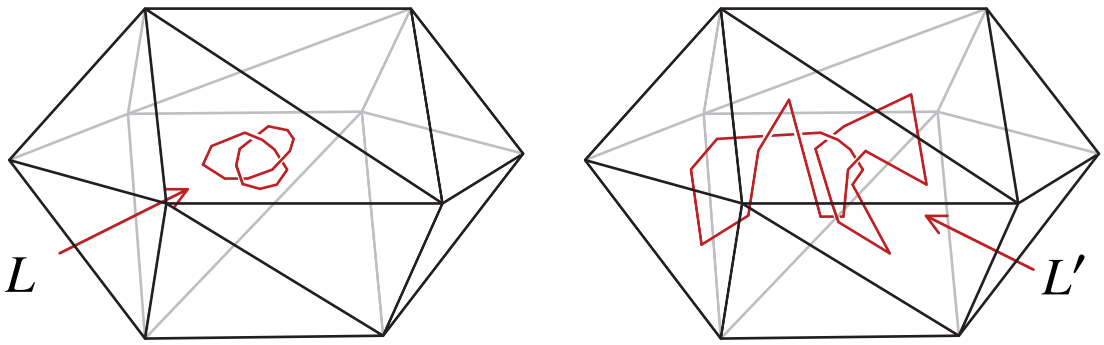
$|T| \xrightarrow{\;h\;} |T'|$

$L \longmapsto L'$

Straight arc $\longmapsto$ Concatenation of $\leq 4^N$ straight arcs

Also insist that $h|_{\partial B} = \mathrm{id}$.

Setup for the main theorem:



$T \dashrightarrow T'$ (N Pachner moves)

$|T| \xrightarrow{\ h\ } |T'|$

$L \longmapsto L'$

Straight arc $\longmapsto$ Concatenation of $\leq 4^N$ straight arcs

Also insist that $h|_{\partial B} = \mathrm{id}$.

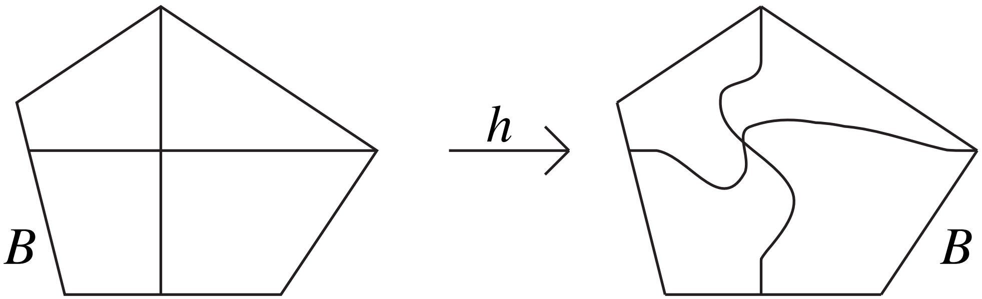
So apply Alexander's trick.

Alexander's trick

In general take B, a convex ball. Let $h : B \to B$ be a homeomorphism such that $h_{\partial B} = \mathrm{id}$.

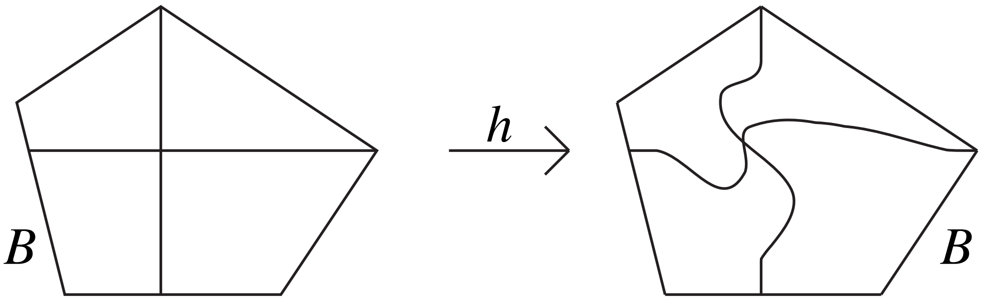
Alexander's trick

In general take B, a convex ball. Let $h : B \to B$ be a homeomorphism such that $h_{\partial B} = \mathrm{id}$.



Alexander's trick

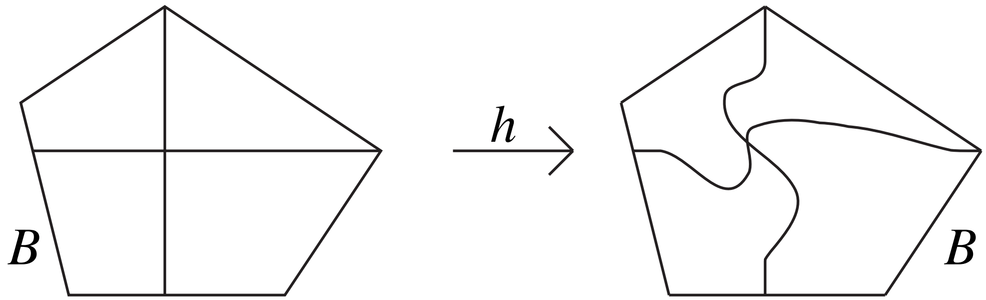
In general take B, a convex ball. Let $h : B \to B$ be a homeomorphism such that $h_{\partial B} = \mathrm{id}$.



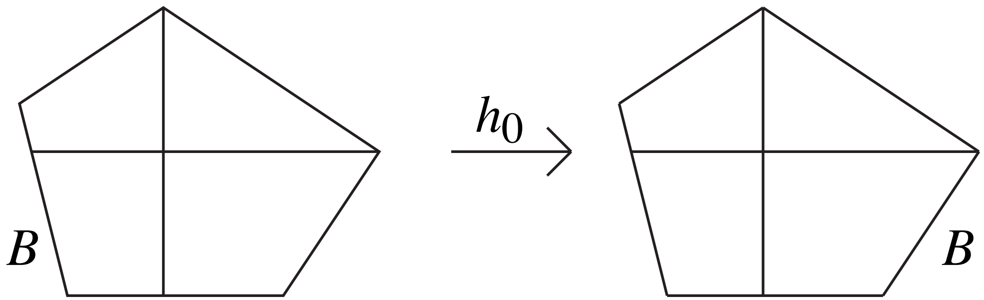
Want an isotopy h_t such that $h_0 = \mathrm{id}$ and $h_1 = h$.

Alexander's trick

In general take B, a convex ball. Let $h : B \to B$ be a homeomorphism such that $h_{\partial B} = \text{id}$.

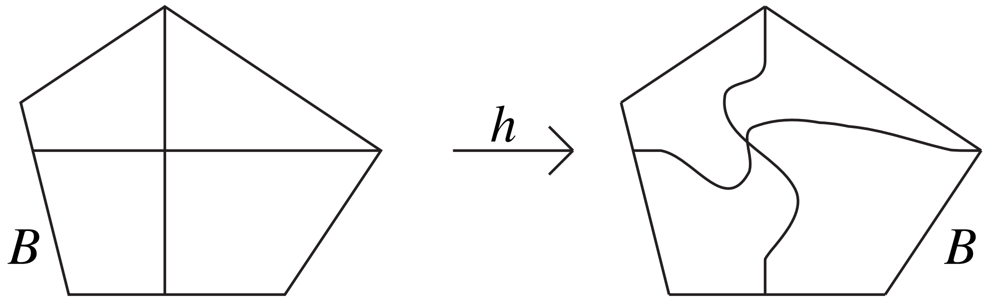


Want an isotopy h_t such that $h_0 = \text{id}$ and $h_1 = h$.

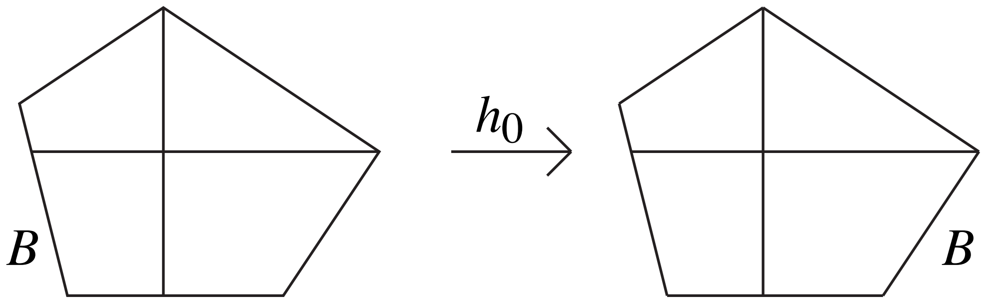


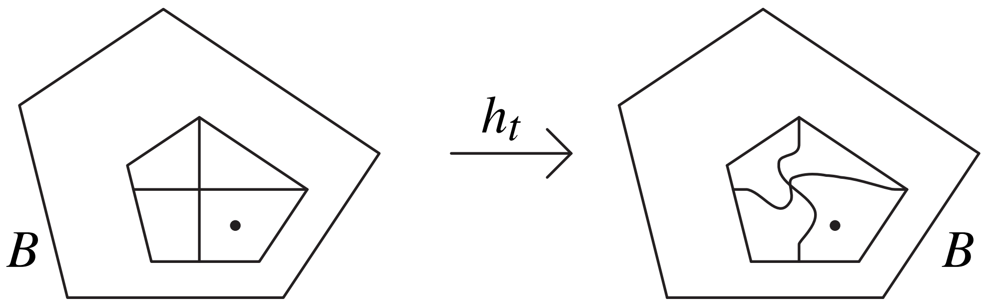
Alexander's trick

In general take B, a convex ball. Let $h : B \to B$ be a homeomorphism such that $h_{\partial B} = \text{id}$.

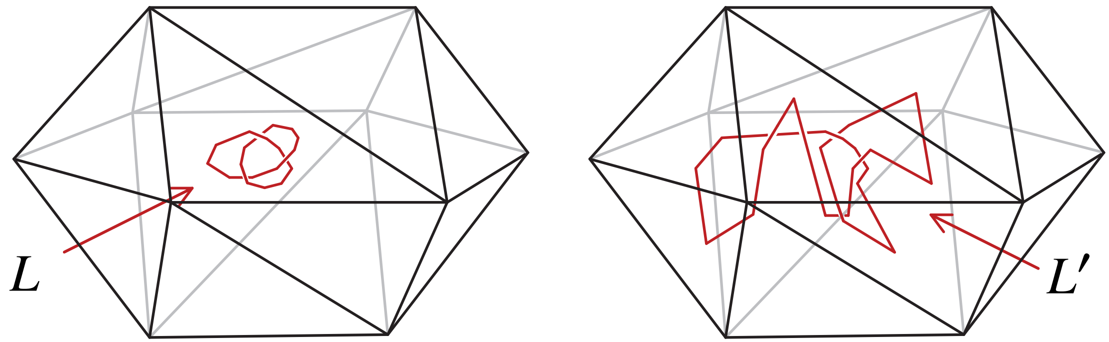


Want an isotopy h_t such that $h_0 = \text{id}$ and $h_1 = h$.

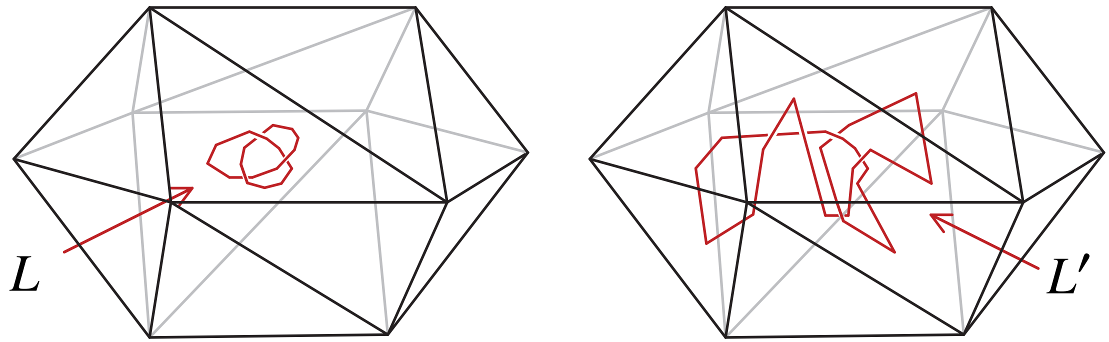




Apply to (B, L):

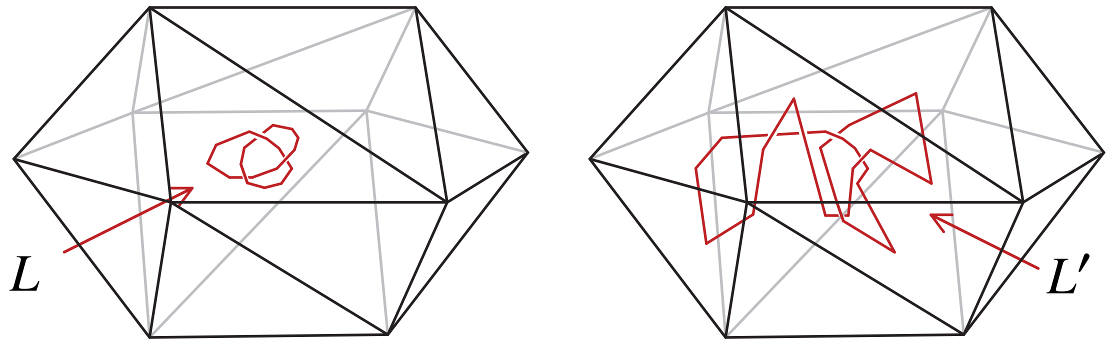


Apply to (B, L):



$h_t(L)$ consists of at most $10,000n4^N$ straight arcs.

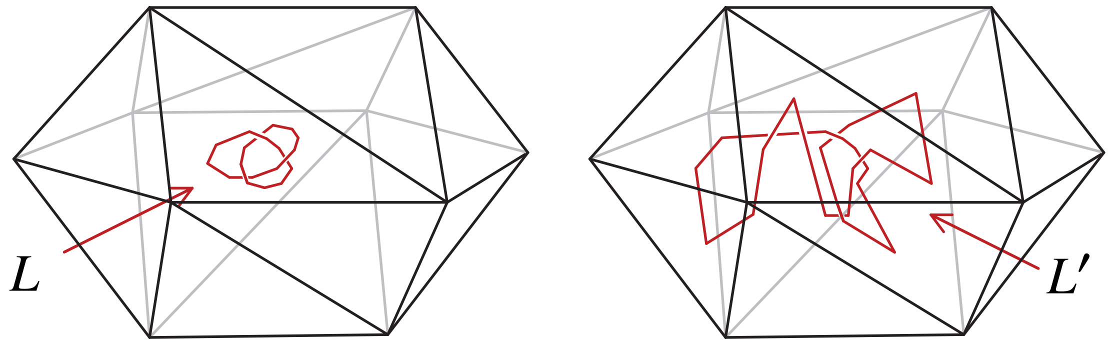
Apply to (B, L):



$h_t(L)$ consists of at most $10,000n4^N$ straight arcs.

$\rightsquigarrow$ upper bound on the number of crossings in an intermediate diagram.

Apply to (B, L):

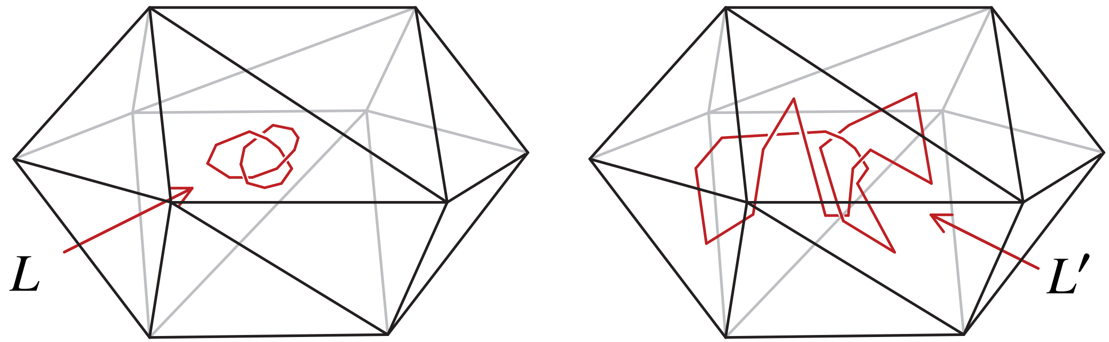


$h_t(L)$ consists of at most $10,000n4^N$ straight arcs.

$\rightsquigarrow$ upper bound on the number of crossings in an intermediate diagram.

$\rightsquigarrow$ bound on number of Reidemeister moves.

Apply to (B, L):



$h_t(L)$ consists of at most $10,000n4^N$ straight arcs.

⇝ upper bound on the number of crossings in an intermediate diagram.

⇝ bound on number of Reidemeister moves.

THE END