RESEARCH STATEMENT

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CONTENTS

1. INTRODUCTION

I am interested in 3-dimensional topology and geometry. These are branches of pure mathematics that deal with the study of shape and space. They draw as their inspiration the real universe that we see around us, and they repay this debt by adding to the body of understanding that we have about the universe. For example, on a large scale topology and geometry have had a profound impact on modern physics, providing powerful tools to expand our understanding of the shape of the universe. On a smaller scale, tools from 3-dimensional topology and geometry are crucial to understanding knots and links. These knots and links arise frequently in nature as long thin molecules, and of particular importance are DNA molecules whose knotting and linking is crucial to understanding gene replication and expression. In my research I like to bring together ideas from diverse fields including geometry, topology, combinatorics and algebra to prove theorems about knots and 3-dimensional spaces.

The following pages contain a summary of my past and and in-preparation work in Sections 2 and 3, and a short selection of current projects in Section 4.

2. Completed projects

2.1. **Ordering the Reidemeister moves of a classical knot.** Published in *Algebraic and Geometric Topology* **6** (2006), 659-671.

One of the most fundamental facts about knots is that any two diagrams of the same knot may be joined by a sequence of Reidemeister moves, as shown in Figure 1. This prompts the question of whether the moves may be carried out so that they are sorted by type. In [4] I answered this question in the affirmative with the following theorem:

Theorem 2.1 (Theorem 1, [4])**.** *Given two diagrams D_1 and D_2 for a link L, D_1 may be turned into D_2 by a sequence of $\Omega_1^{\uparrow}$ moves, followed by a sequence of $\Omega_2^{\uparrow}$ moves, followed by a sequence of Ω_3 moves, followed by sequence of $\Omega_2^{\downarrow}$ moves.*

Furthermore, if D_1 and D_2 are diagrams of a link where the winding number and framing of each component is the same in each diagram, then D_1 may be turned into D_2 by a sequence of $\Omega_2^{\uparrow}$ moves, followed by a sequence of Ω_3 moves, followed by a sequence of $\Omega_2^{\downarrow}$ moves.

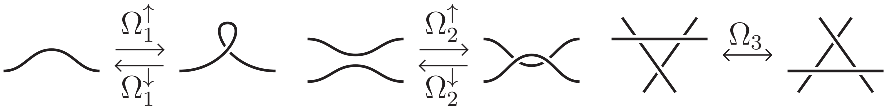

Figure 1. Reidemeister moves

One might expect a proof of this theorem to be rather technical, with many cases to consider. This however turns out not to be the case, and the main new tool in the proof is the use of commutativity diagrams with knot diagrams at each vertex. In the same way that commutativity diagrams in algebraic topology can encode a great deal of algebraic complexity, the commutativity diagrams here encode the complications of knot diagrams. For more details see [4].

In a related project, my student Julian Gold has recently proved the following theorem:

Theorem 2.2 (Theorem 1, [18])**.** *Let D_1 and D_2 be diagrams for the same link that are joined by a sequence of M Reidemeister moves. Let $N = 6^{M+2}M$. Then there exists a sequence of no more than $exp^{(N)}(N)$ moves from D_1 to D_2 ordered in the following way: first $\Omega_1^{\uparrow}$, then $\Omega_2^{\uparrow}$, then Ω_3, then $\Omega_2^{\downarrow}$ and finally $\Omega_1^{\downarrow}$.*

Here the function $\exp(x)$ is the exponential function 2^x, and $\exp^{(r)}(x)$ means iterate this function r times. By combining Theorem 2.2 with Theorem 2.5 below, one obtains an upper bound of the number of sorted moves required to interchange two diagrams of the same link, as a function of the number of crossings in the two diagrams.

Theorem 2.2 was proved as part of Julian Gold's work towards a senior thesis at UC Davis under my supervision. It was partially supported by a VIGRE grant from the NSF.

2.2. **Unknotting genus one knots.** Joint with Marc Lackenby. Published in *Commentarii Mathematici Helvetici* **86** (2011), 383-399.

The unknotting number of a knot is one of the most natural and important measures of a knots complexity, yet it is still mysterious. There is no known algorithm for determining whether a knot has unknotting number one, practical or otherwise. Furthermore, if a knot has unknotting number one, it is not known in general if there are only finitely many ways to unknot it, and if so, whether one can one find them. Answers to these questions would be of interest not just to mathematicians, but also to biologists because of important implications for certain types of site-specific recombination of DNA molecules. In [10] Marc Lackenby and I, building on work of Scharlemann and Thompson [30], answered all the above questions for genus one knots:

Theorem 3 (Theorem 1.1, [10]) *Suppose that K is a knot with genus one and unknotting number one. Then, if K is not the figure-eight knot, there is precisely one crossing change that unknots K, up to equivalence. If K is the figure-eight knot, then there are precisely two unknotting crossing changes, up to equivalence.*

It is possible to be completely explicit about the crossing changes described above. For the figure-eight knot the two unknotting crossing changes are illustrated on the right of Figure 2. The unique crossing change required to unknot the other genus one knots with unknotting number one is illustrated on the left of Figure 2.

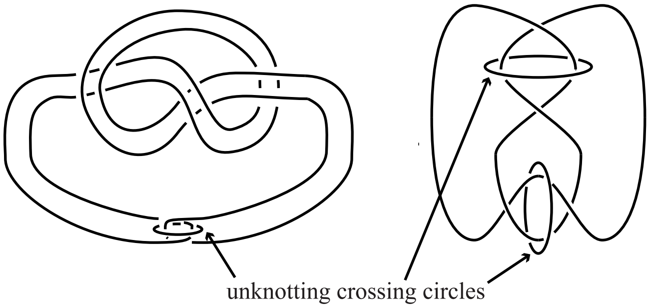

FIGURE 2. Unknotting crossing changes

2.3. **Algorithmically detecting the bridge number of hyperbolic knots.** Submitted.

One of the most natural and widely studied knot invarients is bridge number. However, in common with other natural knot invariants such as unknotting number, calculating the bridge number of specific knots can be difficult in practice. In [5] I proved that it is algorithmically decidable for hyperbolic knots.

Theorem 2.3. *Let K be a hyperbolic knot in S^3. Let M be the exterior of K in S^3. Then, up to ambient isotopy, there are only finitely many bridge punctured 2-spheres* *for M of given Euler characteristic. Furthermore there is an algorithm to find all of these surfaces.*

Corollary 2.4. *There exists an algorithm to determine the bridge number of a hyperbolic knot in S^3.*

The algorithm of Theorem 3.1 has as input any diagram for K and an integer n, and produces as output a finite list of all bridge punctured 2-spheres for M of Euler characteristic n.

2.4. **An upper bound on Reidemeister moves.** Joint with Marc Lackenby. Submitted.

Since Reidemeister's seminal paper [27] in 1926, it has been speculated as to whether there is an explicit upper bound for the number of Reidemeister moves needed to pass between two diagrams for he same knot, as a function of the number of crossings in the two diagrams. See [28] or page 15 of [1] for example. In a celebrated paper [12], Hass and Lagarias provided such a bound when the link in question is the unknot. In [11] Marc Lackenby and I answer the general question with the following theorem, which applies to all knots and links.

Theorem 2.5. *Let D_1 and D_2 be connected diagrams for some knot or link in $\mathbb{R}^3$, and let n be the sum of their crossing numbers. Then D_2 may be obtained from D_1 by a sequence of at most $\exp^{(c^n)}(n)$ Reidemeister moves, where $c = 10^{1{,}000{,}000}$.*

Theorem 2.5 applies to link diagrams with or without orientation. The function $\exp(x)$ is the exponential function 2^x, and $\exp^{(r)}(x)$ means iterate this function r times. Thus, $\exp^{(c^n)}(n)$ is shorthand for a tower of 2s with an n at the top, the height of the tower being c^n.

3. Papers in Preparation

3.1. **Topological and geometric knot theory are not equivalent.** Joint with Joel Hass.

Knot theory studies one-dimensional knots and links in $\mathbb{R}^3$. These are often described as loops of string, or rope, with their ends glued together. Real ropes however are not one dimensional, but have a finite thickness. Indeed, most applications of knot theory to study applications are related more closely to the theory of knots of fixed thickness than to classical knot theory. For example, biologists are interested in curves of fixed thickness as a model for molecules such as DNA and proteins. For these applications the thickness of the knot plays an essential role in determining the possible configurations.

The theory of the knotting and linking of ropes of fixed thickness is clearly different from that of one-dimensional curves, but this difference is difficult to prove. In [7] we give the first rigorous proof that the isotopy class of a three-dimensional knot or link of fixed thickness and fixed length can be distinct from the classical 1-dimensional isotopy class. We thus show for the first time that the theory of knots and links based on ropes of fixed length and thickness is different from the classical theory of knots and links.

The two simplest problems of this type are to construct a *Gordian Unknot* and a *Gordian Split Link*. A Gordian Unknot is a loop of fixed thickness that can be unknotted if it is allowed to stretch, but cannot be unknotted while kept at a fixed length. A Gordian Split Link is a pair of loops of fixed thickness that can be split, that is separated so that each component moves into one of a pair of disjoint convex balls, if each is allowed to stretch, but cannot be split if the lengths of each curve are kept fixed. In this paper we establish, by means of the following theorem, the existence of a Gordian Split Link. Our proof is entirely constructive, and an illustration of the link we use for Theorem 3.1 is shown in Figure 3.

Theorem 3.1. *There exists a Gordian Split Link.*

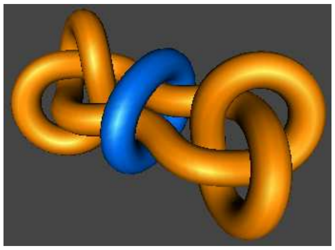

FIGURE 3. A Gordian Split Link

3.2. Unknotting crossing changes and circular Heegaard splittings.

Twenty years ago Scharlemann and Thompson used deep results from sutured manifold theory to prove that a genus reducing crossing change on a knot maybe be realized as untwisting a Hopf band plumbed onto a minimal genus Seifert surface. This gives a hint that understanding genus reducing crossing changes is closely related to understanding how a compact surface in S^3 changes when it is twisted. In [6] I use modern technology from the theory of Heegaard splittings to prove that understanding when two surfaces are related by a single twist implies the existence of an algorithm to determine when two (hyperbolic or fibered) knots of different genus are related by a single crossing change.

Theorem 3.2. *Let K and K' be oriented knots in S^3 where $g(K) > g(K')$ and K and K' are both either hyperbolic or fibered. Then, up to ambient isotopy fixing K (resp. K'), there are finite lists of oriented spanning surfaces $\{S_1, \dots, S_n\}$ for K and $\{S'_1, \dots, S'_{n'}\}$ for K' with the property that if K and K' are related by a single crossing change, then some S_i and some S'_j are related by a single twist.*

Furthermore, there is an algorithm that will take diagrams for K and K' as input, and output such finite lists of spanning surfaces.

Corollary 3.3. *If there is an algorithm to determine whether two compact, oriented surfaces in S^3, each with a single boundary component, are related by a single twist, up to ambient isotopy, then there is an algorithm to determine whether two knots satisfying the hypotheses of Theorem 1 are related by a single crossing change.*

3.3. **Level-n normal surface theory I: The closed case.** Joint with Joel Hass.

The theory of normal surfaces orginated with Kneser [19]. It has been the main algorithmic tool in the study of knots and 3-manifolds since Haken's pioneering work in the 1960s. Normal theory has since been extensively developed, notably by Jaco and Rubinstein.

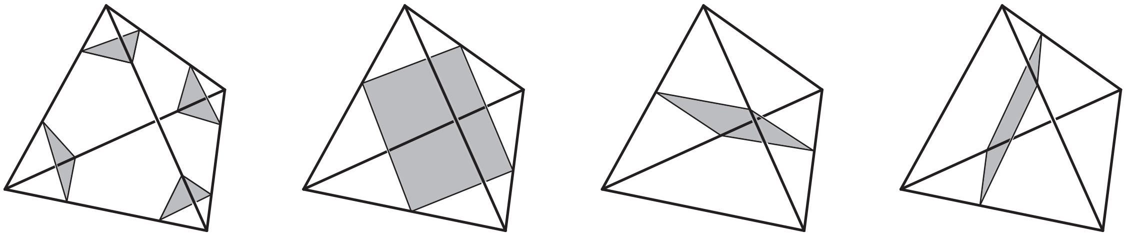

FIGURE 4. The seven types of elementary disc allowed in a normal surface.

In the language we define in [8], standard normal surfaces are level-1 in a series of normal surfaces. For $k > 1$ a level-k normal surface is a standard normal surface that lies in the complement of a level-$(k-1)$ normal surface. The interesting feature of level-k surfaces is that, just as in the case of level-1, they are described by a collection of at most $6t$ integer linear equations of the form $x_i + x_j = x_k + x_l$. Despite its simplicity, this idea has powerful consequences. Using it we can recover many of the advantages of zero-efficient normal surfaces and vertex normal surfaces. Further we can normalize hierarchies of surfaces in a 3-manifold without retriangulating and without the resulting exponential increase in computational complexity.

In [8] we introduce the theory of level-n normal surface theory in the context of closed surfaces in knot complements, allowing for a simplified exposition. We apply it to give simple constructions of the algorithms of Haken and Schubert for UNKNOTTING and SPLITTING. We also give a simple argument showing the existence of a vertex normal surface for these problems, a result originally obtained by Jaco and Tollefson [15]. In a second paper we will consider the general case of hierarchies of surfaces in an arbitrary 3-manifold. See Section 4.3

4. WORK IN PROGRESS

4.1. **Reidemeister moves on alternating links.** Joint with Marc Lackenby.

It is one of the founding theorems of knot theory that any two diagrams of the same knot or link may be joined by a sequence of Reidemeister moves. In [11] we showed that there is an explicit upper bound on the number of moves required in terms of the number of crossings in the two diagrams.

Theorem 4.1 (Theorem 1.1, [11])**.** *Let D_1 and D_2 be connected diagrams for some knot or link in $\mathbb{R}^3$, and let n be the sum of their crossing numbers. Then D_2 may be obtained from D_1 by a sequence of at most $\exp^{(c^n)}(n)$ Reidemeister moves, where $c = 10^{1{,}000{,}000}$.*

Here the function $\exp(x)$ is the exponential function 2^x, and $\exp^{(r)}(n)$ means iterate this function r times. Thus, $\exp^{(c^n)}(n)$ is shorthand for a tower of 2s with an n at the top, the height of the tower being c^n. Clearly this bound is very large indeed, making it natural to ask if one can improve it for particular classes of knot or link. If the link in question is the unknot then this is dramatically so, with only a single exponential being required. This is because of the following theorem of Hass and Lagarias.

Theorem 4.2 (Theorem 1.1, [12]). *Any unknot diagram with n crossings can be transformed to the trivial unknot diagram using at most c^n Reidemeister moves where $c = 2^{10^{11}}$.*

Our goal for this project is to prove the following conjecture.

Conjecture 4.3. *There is a universal constant d so that any n crossing diagram for an alternating knot or link may be transformed to an alternating diagram using at most d^n Reidemeister moves.*

Our plan for proving Conjecture 4.3 is, in essence, as follows: Construct a 2-complex of exponential complexity along which one can slide any diagram of an alternating link to an alternating diagram.

We now discuss this in more detail. Start with any diagram, D, of an alternating link. We wish to bring 3-dimensional technology to bear so begin by realizing D in $\mathbb{R}^3$ with a theorem along the lines of the following:

Theorem 4.4 (Theorem 8.1, [12]). *Given a link diagram D with n crossings, one can construct a triangulated convex polyhedron P in $\mathbb{R}^3$ such that:*

1. *The triangulation contains at most $840n$ tetrahedra.*
2. *Every vertex in the triangulation is a lattice point $(x, y, z) \in \mathbb{Z}^3$, with $0 \leq x \leq 30n$, $0 \leq y \leq 30n$ and $-6 \leq z \leq 6$.*
3. *There is a link L embedded in the 1-skeleton of the triangulation which lies entirely in the interior of P, and which gives D when projected orthogonally onto the plane $z = 0$.*

If L happened to be an unknot then we could, notwithstanding a number of technical points, use normal surface theory to arrange for the disc it bounds to consist of $2^{O(n)}$ flat simplices. We may then slide L across this disc until it projects to the unknot in a way that induces at most $2^{O(n)}$ Reidemeister moves. This is the approach taken by Hass and Lagarias in [12], although in practice there are a variety of issues that need to be addressed with some care, such as how one translates from working in a pair (P, L) to working in the exterior of L. We shall not dwell on these details in the outline that follows.

Let D' be an alternating diagram for L. Let S be one of the checkerboard surfaces. By standard results in normal surface theory one may arrange for S to be normal in P and consist of $2^{O(n)}$ normal discs. The surface S to a large extent encodes the diagram D', however it does not provide an explicit recipe on its own for simplifying D. To this end we consider the regions of the diagram D' which do not form part of the checkerboard surface S. We wish to normalize these disks, which we collectively call

F, with an estimable number of normal discs as well. One way to proceed would be to re-triangulate P to make S lie in the 2-skeleton and then apply normal surface theory in the 3-manifold with boundary pattern obtained by cutting along S. After making F normal with an estimable number of normal discs we then consider the union of S and F which is a 2-complex, C, which roughly agrees with the plane of the diagram D'. We could then, modulo some important details, slide L along C until it projects to D'. This process of sliding across C induces a sequence of Reidemeister moves from D to D' under projection, and the number of these moves will be on the order of the number of normal discs in $S \cup F$. Now, re-triangulating increases the number of tetrahedra in P exponentially. Further the bound on the number of normal discs in F is exponential in the number of tetrahedra in the re-triangulated triangulation. Hence the normal surface F constructed in this way would be expected to have $\exp^{(2)}(O(n))$ normal discs, and so we would expect the sliding process to induce double exponentially many Reidmeister moves.

This raises the question of how we plan to obtain the single exponential bound of Conjecture 4.3. As exposed above, the key challenge is to bound the number of flat simplices in F by $2^{O(n)}$, saving an exponent. To achieve this we will not re-triangulate to make S lie in the 2-skeleton, but rather cut along S and work in the remnants of the original triangulation of P. This still has exponentially many pieces, but crucially all but linearly many of these pieces glue together to form I-bundles which each look like an I-bundle over a patch of normal surface. We will now apply a version of normal surface theory in this new setting to arrange for F to consist of exponentially many pieces, each of which consists of at most exponentially many flat simplices. Because $(2^{O(n)})^2 = 2^{O(n)}$ we expect to obtain the desired upper bound on the number of flat simplicies in $S \cup F$.

4.2. **Unknotting with Reidemeister moves.** Joint with Joel Hass and Marc Lackenby.

As mentioned in the previous section, it is a theorem of Hass and Lagarias [12] that the number of Reidemeister moves required to pass from an unknot diagram to the trivial diagram is bounded above by an exponential function of the number of crossings in the initial diagram. The proof of this fact uses normal surface theory to arrange the disc that the unknot bounds so that it consists of an exponential number of flat simplices. One then one uses this to disc as a recipe for retracting the knot so that it projects to a trivial unknot diagram. The bound on the number of Reidemeister moves that this induces comes from the bound on the number of simplices of the disc. Now, it is known that there are sequences of PL embeddings of the unknot for whom the minimal number of flat simplices required for a spanning disc grows exponentially with the number of edges of the embedding. This is a result of the following theorem which appears in [14].

Theorem 4.5. *There exists a sequence of knots in $\mathbb{R}^3$, denoted K_n, such that for each $n \in \mathbb{N}$, K_n is an unknotted polygon with at most $10n + 9$ edges but for which any PL* *embedding of a triangulated disc into* $\mathbb{R}^3$ *with boundary* K_n *consists of at least* 2^{n-1} *triangular faces.*

Thus there can be no material improvement on the Hass-Lagarias bound if one only attempts to reduce the number of simplices in the spanning disc. Our goal is an ambitious one: to exhibit a polynomial upper bound on the number of Reidemeister moves required for unknotting.

Conjecture 4.6. *There exists universal constants* A *and* k *such that any unknot diagram with* n *crossings may be changed into the trivial diagram using at most* An^k *Reidemeister moves.*

Our approach is to look for parts of the spanning disc that consist of many parallel sheets, using ideas from normal surface theory, and to simplify all of these sheets at once. While this is an appealing idea, the analysis can quickly become complicated, and other people have thought about this approach. The key new idea we bring to this old problem is to recognize that the setup in which one works can make the analysis easier. In particular ideas from thin position and braid theory seem to be helpful. This was also true in the proof of Theorem 4.5. We now discuss our approach in more detail.

Let D be an unknot diagram. Regard D as lying in the $x-y$ plane in $\mathbb{R}^3$. Consider the height function $h : \mathbb{R}^3 \to \mathbb{R}$ given by $h(x, y, z) = y$. Using a polynomial number of Reidemeister moves we may stretch the diagram so that it is in bridge position with respect to h. Now consider the disc, S, which spans the unknot and also consider a plane, P, perpendicular to the y-axis. Suppose that P hits the knot in a maximal number of points. The intersection of P with S consists of arcs and simple closed curves. It is by analyzing the possible configurations of these arcs and curves, paying particular attention to parallel regions that we aim to exhibit a collection of moves that will simplify the unknot very rapidly. At intermediate stages of the simplification process the unknot might no longer sit in bridge position with respect to h since some of the moves we will be applying can introduce new thick and thin level sets. However each of these will in turn contain an arrangement of arcs and curves of intersection with S which may be analyzed in a similar fashion.

If we are successful it will provide a conceptually easy proof that the unknotting problem resides in **NP**, a fact first proved by Hass, Lagarias and Pippenger [13]. We will also consider the number of moves required to pass from a split link diagram to a diagram that is split.

4.3. **Level-n normal surface theory II.** Joint project with Joel Hass.

The theory of normal surfaces and their adaptations has been the main tool for constructing algorithms to answer questions about 3-manifolds since Haken's pioneering work in the 1960s. Recognition algorithms for links and Haken 3-manifolds, for example, are based largely on this. More recent times have seen the use of normal surface theory and related ideas for purposes other than exhibiting the existence of algorithms. These other applications fall broadly into three categories: First, the computational complexity of certain decision problems in 3-manifold theory has been investigated.

Second, a variety of non-algorithmic theorems about 3-manifolds have been proved. Third, with the development of more powerful computers and considerable innovation in implementational method, normal surface theory has been used in practice to answer questions about specific 3-manifolds. The following are examples to illustrate each of these themes.

Theorem 4.7 (Theorem 1.1, [13])**.** *The decision problem of determining whether a given knot is ambient isotopic to the unknot lies in the complexity class* **NP**.

Theorem 4.8 (Theorem 1, [20])**.** *The canonical polyhedral decomposition of a hyperbolic once punctured torus bundle over the circle is its monodromy ideal triangulation.*

Theorem 4.9 (Theorem 1.1, [3])**.** *The Seifert-Weber dodecahedral space is non-Haken.*

Thus it is clear that normal surface theory is a powerful tool that rightly lies at the heart of 3-manifold theory. This is essentially because many fundamental questions about 3-manifolds, whether topological, geometric or group-theoretic, are one way or another questions about surfaces. The philosophy behind normal surface theory is to try to understand surfaces in a given 3-manifold, M, by trying to arrange for the surface to interact agreeably with a decomposition for M. This decomposition could be a triangulation, in the classical sense, or something more complicated such as an ideal triangulation, polyhedral decomposition or handle structure. If such a decomposition is chosen to somehow reflect the global topology and/or geometry of M, the canonical polyhedral decomposition being a good example where it applies, one can begin to see how non-algorithmic theorems can arise by considering various adaptations of normal surfaces.

Given the above comments, it makes sense to wonder if objects other than embedded surfaces can be made to interact well with a decomposition of a 3-manifold. Now, various flavors of hierarchy, in particular sutured manifold hierarchies, have been an important tool in 3-manifold theory for many years. However hierarchies, by which we roughly mean decompositions of 3-manifolds realized by cutting along one surface after another, do not interact particularly well with classical normal surface theory. Historically what one tends to do is arrange for the first surface in a hierarchy to be normal, with respect to a triangulation say, then re-triangulate the 3-manifold to make this surface part of the 2-skeleton, then make the second surface normal with respect to the new triangulation, re-triangulate again, and so on. This approach is unsatisfactory for purposes other than demonstrating algorithmic decidability because many of the features of the original triangulation are lost, such as geometric properties or number of tetrahedra. The goal of this project is to develop a theory of *level-n normal surfaces*, a generalization of normal surface theory, for whom no re-triangulating is necessary and which interacts naturally with 3-manifold hierarchies, in their loosest sense. We expect this technology to have a great many applications in 3-manifold theory distributed right across the three themes described at the start of this section.

Let X be a polyhedron. We say that a disc D properly embedded in X is a *normal disc* if it does not intersect the vertices of X and intersects the edges of X transversely in at most one point along each edge. Let M be a compact 3-manifold. Let M have a polyhedral decomposition, P. We say that a properly embedded surface S in M is *normal* with respect to P if it intersects each polyhedron of P in a finite collection of normal discs. It is straightforward to develop a theory of normal surfaces in polyhedral decompositions which closely follows the theory of normal surfaces in triangulations. We shall write $M - S$ for the 3-manifold obtained by cutting M along S. Note that $M - S$ inherits a polyhedral decomposition.

For a separating normal surface S in M with respect to P, each polyhedron of P is decomposed into pieces of two types. The first type lies between two parallel copies of the same type of normal disc. We call these *product pieces*. The other pieces are called *core pieces*. The product pieces have two types of boundary, namely that coming from the boundary of the ambient polyhedron, which we call the *vertical boundary*, and that coming from S, which we call the *horizontal boundary*. If one glues together the product pieces along their vertical boundaries, as specified by how they sit in M, they form a collection of I-bundles which we call *parallelity regions*, following Lackenby [21]. We say that a surface properly embedded in a parallelity region is *horizontal* if it intersects each fiber of the I-bundle structure exactly once. We say that the ordered pair (S, F) is a *level*-2 *normal surface* with respect to P if F is properly embedded in $M - S$, it intersects each core piece of $M - S$ in a collection of normal discs and each parallelity region of $M - S$ in a collection of horizontal surfaces. The definition of a level-n normal surface is analogous to that of a level-2 normal surface, although in practice there are several technical points that need to be addressed with some care, such as how one handles surfaces with non-empty boundary on opposite sides of a previous surface in the hierarchy. We have chosen to omit these technicalities from this proposal.

Despite the fact that we are omitting some important details from the definition of a level-n normal surface in a 3-manifold M, we are in a position to see some of their agreeable features. Perhaps the most important point is that to describe a level-n normal surface one requires a sequence of n vectors, $(v_1, \ldots, v_n)$. The vector v_1 describes the first surface in the hierarchy using standard normal surface coordinates. The second vector, v_2, describes a surface in coordinates whose meaning depends on v_1; there is one coordinate for each parallelity region and each core piece arising from cutting along a normal surface corresponding to v_1. The third vector, v_3, describes a surface in coordinates whose meaning depends on both v_1 and v_2, and so on. This brings us on to an important point, which is that the number of coordinates required for v_i grows linearly with i, irrespective of how complicated the previous surfaces are. If one of the surfaces, v_i, in a level-n normal surface is very complicated indeed then that complexity is manifested in the complexity of the horizontal pieces in later surfaces, but not their number.

We now turn our attention to possible applications of level-n normal surfaces. We give four examples. First the project detailed in Section 4.1, to efficiently bound the number of Reidemeister moves required to pass from a diagram of an alternating link to an alternating diagram could be phased in the language of level-n normal surfaces. Our second example comes from the algorithmic side of normal surface theory. As is well known to experts, one of the key technical obstructions to using normal surface theory in practice is that fundamental surfaces of non-negative Euler characteristic can arise with great abundance in any given 3-manifold. In particular, efforts to control the prevalence of normal 2-spheres has led to the development of 0-efficient triangulations. The following theorem illustrates the extent to which level-n normal surfaces can be used as an alternative to 0-efficient triangulations [17].

Theorem 4.10 (C,Hass)**.** *Let M be a compact orientable 3-manifold with triangulation T. Suppose M contains no embedded projective plane. If M contains an essential 2-sphere then it contains a level-n normal surface $(S_1, \dots, S_n)$ where each S_i is a 2-sphere, each S_i is fundamental, precisely one S_i, namely S_n, is essential in M and n is at most 20 times the number of tetrahedra in T.*

When we say that each S_i is fundamental, we mean that it is fundamental with respect to the coordinates inherited from the previous surfaces $S_1, \dots, S_{n-1}$.

Of course, this Theorem follows immediately from Theorem 4.1.12 of [22] where it is proved that, under the hypotheses of Theorem 4.10, there exists a fundamental essential normal 2-sphere in M. That argument, however, is quite technical, relying on a careful analysis of intersection curves of normal surfaces. What is noteworthy about Theorem 4.10 is the fact that its proof requires none of the detailed topological arguments usually required to show that a certain type of surface can be realized as one that is fundamental. This approach extends to questions other than the existence of essential 2-spheres, and when searching for normal surfaces in irreducible 3-manifolds it is possible to effectively discount normal 2-spheres by passing to a suitably chosen level-n normal surface. Thus level-n normal surfaces can be used to glean much of the benefit of 0-efficient triangulations [17]. While this is heartening, it goes only a small way towards controlling all surfaces on non-negative Euler characteristic in a given 3-manifold and it is this that motivates the following question:

Question 4.11. *Is it possible to use the theory of level-n normal surfaces to gain effective control over normal tori in a given compact 3-manifold.*

If the answer to Question 4.11 was 'yes', then it would present an alternative method of obtaining the benefits of 1-efficient triangulations [16], avoiding the technical difficulties associated with these.

Our third example of a possible application is related to the following theorem, proved by me and Marc lackenby in [11].

Theorem 4.12 ([11])**.** *Let M be a compact orientable irreducible 3-manifold with boundary a non-empty collection of tori. Suppose that the closure of each component of the complement of the characteristic submanifold of M satisfies at least one of the following conditions:*

- *it does not fibre over the circle; or*
- *it is not a surface semi-bundle; or*
- *it has at least two boundary components.*

Let $T_{\partial M}$ be any triangulation of ∂M. Then there is a triangulation $\mathcal{T}'_{\mathrm{can}}$ of M with the following properties. Its restriction to ∂M equals $T_{\partial M}$. Further, if T is any triangulation of M with t tetrahedra, such that the restriction of T to ∂M also equals $T_{\partial M}$, then there is a sequence of at most $\exp^{(a^t)}(t)$ interior Pachner moves, followed by a homeomorphism of M that is the identity on ∂M, taking T to $\mathcal{T}'_{\mathrm{can}}$. Here, $a \leq 2^{162}$.

It is this theorem that underpins our bound on Reidemeister moves in Theorem 2.5. Theorem 4.12 is based on work of Aleksandar Mijatović, who proved slightly weaker results in a series of papers [23, 24, 25, 26]. It is the largeness of these bounds that cause our upper bound on Riedemeister moves to be so large. This provides motivation to improve the upper bound of Theorem 4.12. To achieve this we need to modify the proof of Theoem 4.12, which considers in a given triangulation, T, something akin to a normal hierarchy for the 3-manifold in question, and cones in each complementary 3-ball. These 'hierarchies' have exponential depth in terms of the number of tetrahedra in T. Furthermore, at each stage of constructing the 'hierarchy' one re-triangulates, increasing the number of tetrahedra exponentially. Thus one obtains a tower of exponentials of exponential height. If instead one uses level-n normal surfaces then one should be able to obtain a significantly improved bound, implying a significantly improved bound on the number of Reidemeister moves required to interpolate between two link diagrams. It is not yet clear exactly how big this bound will be, but a good starting point would be to aim for a tower of exponentials of fixed height. Thus we make the following conjecture which, despite superficial similarity to Theorem 4.1, would represent a very significant improvement.

Conjecture 4.13. *There exists a universal constant c such that for any two connected diagrams D_1 and D_2 of the same knot or link in $\mathbb{R}^3$, D_2 may be obtained from D_1 by a sequence of at most $\exp^{(c)}(n)$ Reidemeister moves, where n is the sum of the crossing numbers of D_1 and D_2.*

Our fourth example of a potential application of level-n normal surfaces is related to the third. We make the following conjecture.

Conjecture 4.14. *The decision problem of determining whether two given triangulated 3-manifolds satisfying the conditions of Theorem 4.12 are homeomorphic lies in the complexity class* **NP**.

To prove this conjecture we would seek to encode hierarchies used in the proof of Theorem 4.12 by sequences of vectors, $(v_1, \ldots, v_n)$. Then, given two such vectors, inspect them in polynomial time to see if they come from homeomorphic 3-manifolds. Thus, it is this pair of vectors that we would seek to show constituted a polynomial time certificate for this decision problem. A corollary of Conjecture 4.14 would be that the recognition problem for links in S^3 lies in the complexity class **NP**.

We conclude this section by emphasizing that one could have described each of these applications as an individual project. However it is our view that the mathematical community will be better served by first establishing a unified theory to provide context for these specific projects. We will present an account of level-n normal surface theory in the case where all the surfaces are closed in an expository paper [8] (see Section 3.3), currently in the late stages of preparation. Further details will appear in a longer paper [9] which deals with general hierarchies.

4.4. **Intersection numbers of 3-manifolds.** Joint with Matthew Rathbun.

We define the intersection number of a compact orientable 3-manifold M, which we denote by $i(M)$, to be the minimal number of intersections in any diagram for a minimal genus Heegaard splitting of M. This may be regarded as being somewhat analogous to crossing number for knots. It is a longstanding open problem to determine whether crossing number is additive under connect sum of knots. This inspires us to investigate the behavior of intersection number under 3-manifold connect sum, and to make the following conjecture.

Conjecture 4.15. *There exists a universal constant C such that any pair of compact orientable 3-manifolds M_1 and M_2 satisfy*

$$\frac{1}{C}(i(M_1)+i(M_2)) \leq i(M_1\#M_2) \leq i(M_1)+i(M_2)$$

where $M_1\#M_2$ denotes the connect sum of M_1 and M_2.

The following theorem was proved by Marc Lackenby [21], and may be regarded as being in the same spirit as Conjecture 4.15.

Theorem 4.16. *Let $K_1,\ldots,K_n$ be oriented knots in the 3-sphere. Then*

$$\frac{1}{152}(c(K_1)+\ldots+c(K_n)) \leq c(K_1\#\ldots\#K_n) \leq c(K_1)+...+c(K_n).$$

In Theorem 4.16 $c(K)$ denotes the crossing number of a knot K, and $\#$ denotes connect sum.

One might regard Conjecture 4.15 as being a little cautious, and a bolder conjecture would be that $i(M_1\#M_2) = i(M_1)+i(M_2)$. This would be analogous to the additivity of crossing number of knots. Both these bolder conjectures of strict additivity seem to be at least an order of magnitude more difficult.

The approach Lackenby adopted in [21] to prove Theorem 4.16 used a version of normal surface theory in a handle structure coming from a knot diagram to reconstruct diagrams of $K_1,\ldots,K_n$ starting with a diagram of $K_1\#\ldots\#K_n$. On close inspection this approach does not seem to lend itself well to Conjecture 4.15, and the line of attack we adopt in this project is very different.

The following theorem, which was conjectured by Cameron Gordon, was proved recently by Qiu and Scharlemann [29], and also independently by Bachman using different methods [2].

Theorem 4.17 ('The Gordon Conjecture')**.** *The connect sum of two Heegaard spittings is stabilized if and only if one of the summands is stabilized.*

Our approach to Conjecture 4.15 is to extend the methods used in [29]. For this reason we briefly expose the methods used in that paper.

Let M be the connect sum of two closed orientable 3-manifolds, M_1 and M_2. Let H_1 and H_2 be respective Heegaard surfaces for M_1 and M_2. Let H be the Heegaard surface for M obtained by taking the connect sum of M_1 and M_2 in such a way that, for $i = 1, 2$, the 3-ball removed from M_i intersects H_i in in single disc, and when the resulting punctured 3-manifolds are glued, the resulting upper and lower hemispheres agree. This yields a Heegaard surface for M, which we call H, and which intersects the

2-sphere along which we glued, S say, in a single curve. It is clear that if at least one of H_1 and H_2 is stabilized, then H must also be stabilized. The Gordon Conjecture states that the converse also holds. To understand why this is a hard problem, consider a pair of compression discs for H, A and B, on opposite sides of H. Figure 5 illustrates this, although B, the compression disc to the lower side of H in the figure, is omitted for the sake of clarity. As can readily be seen from the figure, the discs A and B are each cut into several subdiscs by S. Furthermore, exactly one pair of these subdiscs, one coming from of A, a say, and one coming from B, b say, intersect in exactly one point on H, which may be to either side of S. What one would like to say is that a and b give rise to a stabilizing pair for H_1 or H_2. The reason that they do not, or at least not in an obvious way, is that the discs a and b will in general intersect S. Now, when one seeks to recreate M_1 or M_2 from M, one throws away the part of M to the side of S we do not want, and glues the upper an lower hemispheres of S. This gluing will in general introduce more intersections between a and b than the single intersection they had originally.

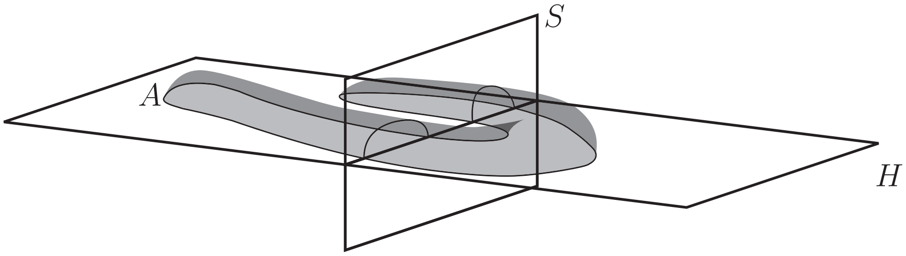

Figure 5

The approach taken in [29] can now be described as follows: Instead of focusing on just one side of S, cut along S and glue the upper and lower hemispheres of S in both resulting 3-manifolds. The two resulting 3-manifolds are precisely M_1 and M_2. The Heegaard surface H induces the Heegaard surfaces H_1 and H_2 after capping off with the discs obtained by gluing the two hemispheres of S, and these discs give subdiscs of both H_1 and H_2. Label these subdiscs P_1 and P_2 respectively. If one keeps track of what happens to A and B throughout this process we see a collection of discs, distributed to each side of H_1 and H_2. This collection has precisely one point of intersection outside P_1 and P_2, but we will in general see a great many intersections inside P_1 and P_2. It is these intersections inside P_1 and P_2 that Qiu and Scharlemann remove, by means of moves such as banding one disc onto another or removing a disc altogether, and furthermore they do so in a way that preserves some the the character of the original collection of discs and in particular does not introduce new intersections outside P_1 and P_2. This is not an easy exercise, and in the course of their argument significant technical difficulties arise which are handled with great skill and ingenuity. For example, the subsurfaces P_i change so that they might not be discs after the outset, although they retain the property that arcs (of the right sort) intersect at most once. We will not seek to reproduce all the details inherent in their argument here, but it is worth emphasizing one important aspect, namely the following: This is an argument that takes place, simultaneously, in *two* ambient 3-manfiolds, not one. Combinatorial information describing the relationship between what is happening in each of these 3-manifolds therefore needs to be kept track of, and this is achieved with the use of abstract trees, decorated with rich information, whose vertices correspond to the discs inherited from A and B and whose edges capture, at least at the outset, the way that they were patched together in M.

We are now in a position to see, at least superficially, how one might wish to use the technology in [29] to approach Conjecture 4.15. In the forthcoming discussion we use the same labeling of objects as above to highlight the similarities, although this is a new discussion. Consider a minimal genus Heegaard diagram for $M = M_1 \# M_2$ with minimal number of intersections, $i(M)$. Further suppose that this is realized by a Heegaard surface H. By Haken's lemma, we may arrange for a reducing 2-sphere, S, separating $M_1 -$ (3-ball) from $M_2 -$ (3-ball) to intersect H in a single curve. The Heegaard diagram for H gives rise to a collection of discs arranged on each side of H. This setup shares many features with the setup for the Gordon conjecture used in [29], the principle differences being that now there are more discs either side of H and that now there are $i(M)$ intersections between these discs rather than just one. Proceed as in [29]: Cut M along S and glue the resulting hemispheres in the resulting 3-manifolds. This yields two 3-manifolds, which are precisely M_1 and M_2. They have respective Heegaard splittings which we call H_1 and H_2, and these are marked with respective subdiscs, P_1 and P_2, coming from S. Further we see a collection of discs each side of H_1 and H_2. These discs cut the handlebodies in which they reside into balls. They have precisely $i(M)$ intersections in total outside of P_1 and P_2 but possibly a great many intersections inside P_1 and P_2. Our goal is to remove these intersections inside P_1 and P_2 without introducing (too many) new intersections outside P_1 and P_2. The resulting discs will be the ones that induce Heegaard diagrams for M_1 and M_2, with corresponding Heegaard surfaces H_1 and H_2. The total number of intersections in both these Heegaard diagrams will be precisely $i(M)$ plus the number of new intersections we have introduced outside P_1 and P_2 plus the number of intersections remaining inside P_1 and P_2.

The remainder of this section will be concerned with providing some more detail in the light of the above discussion. Let us refer to the two sides of H as being *above* H and *below* H. We will refer to the sides of H_1 and H_2 as *above* and *below* according to which side of H they correspond to after taking connect sum. The trees referred to above are really forests of trees, and there is one forest for discs above H_1 and H_2 and another forest for discs below H_1 and H_2. In [29] each forest has precisely one special vertex called a 'distinguished root', and this corresponds to the disc containing the original point of intersection in the stabilizing pair of discs. Very roughly the argument of [29] runs as follows. Use the rich combinatorial information contained in the forests of discs to find a single pair of discs, one above, and one below. This pair of discs is somehow extremal in the sense that the discs are far away in their respective forests from the distinguished roots. We do not know which ambient 3-manifold contains each of these discs, M_1 or M_2, but we do know, for complicated reasons, that *if* they lie in the same ambient 3-manifold (ie they are both in M_1 or both in M_2) *then* they intersect in precisely one point inside P_1 or P_2. Because these discs were chosen so that they were far away from the distinguished root, meaning in particular that neither of them is a distinguished root, this is the only point of intersection between these two discs. Hence we have a stabilizing pair of compression discs for M_1 or M_2 and we can stop if the Gordon Conjecture is our motivation. On the other hand if the two discs are in different ambient 3-manifolds, so that one lies in M_1 and one lies in M_2, we perform a complicated and delicate operation on the whole setup. This operation involves changing the subsurfaces P_1 and P_2, changing the discs by banding them onto each other in a certain way, throwing away some discs altogether, adding new objects called 'op-arcs' which keep track of what has happened in a certain sense and updating the forests above and below as well as the combinatorial information associated with these. Despite the fact that this operation, termed the 'fundamental construction', is very delicate, one can understand the way that it reduces complexity by considering the effect on the forests of discs above and below. Essentially what happens is this: We start with a rooted tree, the root being the distinguished root, and when we perform the fundamental construction we change the tree by removing a vertex and adjusting the discs corresponding to adjacent vertices by banding somehow. For this it is important that the vertex that we removed was not adjacent to the distinguished root, because if it was then we would introduce new intersections outside P_1 and P_2 in the banding of discs that takes place.

We are now in a position to see how Conjecture 4.15 might be proved. Instead of having one distinguished root above and below, corresponding to just one disc above and one disc below, we now have several 'distinguished roots', namely those corresponding to a discs with at least one point of intersection outside P_1 and P_2. Thus there are no more than $i(M)$ of these distinguished roots because otherwise there would be more than $i(M)$ intersections outside P_1 and P_2 in total. We may perform the fundamental construction in a similar way to [29] provided that we are not adjacent to any of the 'distinguished roots'. Thus, so long as we are in the case where the fundamental construction applies, we can reduce complexity in a way that removes discs not adjacent to a distinguished root. The other case to consider is when our extremal pair of discs, one above and one below, lies in the same ambient 3-manifold. In this case the Gordon Conjecture argument of [29] just stops, since we have a stabilizing pair of discs of M_1 or M_2. In our context this case does not arise, because we have assumed all Heegaard splitting to be of minimal genus, implying that they are not stabilized.

If one were going to speculate as to how one might prove strict additivity of $i(M)$, that is $i(M_1 \# M_2) = i(M_1) + i(M_2)$, I suspect that there is an argument, based loosely on the technology above, which uses forests of discs of this sort to specify a helpful sequence of stabilizations and destabilizations to H_1 and H_2. However we are not actively pursuing this line at present because the analysis quickly becomes very complicated.

We conclude this section by remarking briefly that if one defines the intersection number of a Heegaard splitting to be the minimal number of intersections in any diagram for that particular isotopy class of splitting, then understanding the extent to which intersection number increases under destabilization has important implications regarding the topology of finite covers of hyperbolic 3-manifolds. It is in this way that Conjecture 4.15 has implications that lie well outside the theory of Heegaard splittings.

References

[1] Colin Adams, *The knot book. An elementary introduction to the mathematical theory of knots*, W. H. Freeman and Company, New York, 1994.

[2] David Bachman, *Connected sums of unstabilized Heegaard splittings are unstabilized*, Geom. Topol. **12** (2008), no. 4, 2327–2378.

[3] Ben Burton, Hyam Rubinstein, and Stephan Tillman, *The Weber-Seifert dodecahedral space is non-Haken*, Trans. Amer. Math. Soc. (to appear).

[4] Alexander Coward, *Ordering the reidemeister moves of a classical knot*, Algebraic and Geometric Topology **6** (2006), 659–671.

[5] ———, *Algorithmically detecting bridge numbers of hyperbolic knots*, 2007.

[6] ———, *Unknotting crossing changes and circular heegaard splittings*, 2011.

[7] Alexander Coward and Joel Hass, *Topological and geometric knot theory are not equivalent*, 2011.

[8] ———, *Level-n normal surface theory I: The closed case*, in preparation.

[9] ———, *Level-n normal surfaces*, in preparation.

[10] Alexander Coward and Marc Lackenby, *Unknotting genus one knots*, 2010.

[11] ———, *An upper bound for Reidemeister moves*, in preparation.

[12] Joel Hass and Jeffrey Lagarias, *The number of Reidemeister moves needed for unknotting*, J. Amer. Math. Soc. **14** (2001), 399–428.

[13] Joel Hass, Jeffrey Lagarias, and Nicholas Pippenger, *The computational complexity of knot and link problems*, Journal of the ACM **46** (1999), no. 2, 185–211.

[14] Joel Hass, Jack Snoeyink, and William P. Thurston, *The size of spanning disks for polygonal curves*, Discrete and Computational Geometry **29** (2003), 1–17.

[15] William Jaco and Jeffrey L. Tollefson, *Algorithms for the complete decomposition of a closed 3-manifold*, Illinois J. Math. **39** (1995), no. 3, 358–406. MR 1339832 (97a:57014)

[16] William H. Jaco and J. Hyam Rubinstein, *1efficient triangulations of 3-manifolds*, 2002.

[17] ———, *0-efficient triangulations of three-manifolds*, J. Diff. Geometry **65** (2003), 61–168.

[18] Gold Julian, *A bound for orderings of reidemeister moves*, 2011.

[19] H Kneser, *Geschlossene Flachen in dreidimensionalen Mannigfaltigkeite*, Jahr. der Deutsch. Math. **38** (1929), 248–260.

[20] Marc Lackenby, *The canonical decomposition of once-punctured torus bundles*, Comment. Math. Helv. **78** (2003), 363–384.

[21] ———, *The crossing number of composite knots*, J. Topology **2** (2009), 747–768.

[22] Sergei Matveev, *Algorithmic topology and classification of 3-manifolds*, Algorithms and Computation in Mathematics, vol. 9, Springer, 2003.

[23] Aleksandar Mijatović, *Simplifying triangulations of the 3-sphere*, Pacific J. Math. **208** (2003), no. 2, 291–324.

[24] ———, *Triangulations of fibre-free Haken 3-manifolds*, Pacific J. Math. **219** (2003), no. 1, 139–186.

[25] ———, *Triangulations of Seifert fibred manifolds*, Math. Ann. **330** (2004), no. 2, 235–273.

[26] ———, *Simplical structures of knot complements*, Math. Res. Lett. **12** (2005), no. 5-6, 843–856.

[27] Kurt Reidemeister, *Knotten und Gruppen*, Abh. Math. Sem. Univ. Hamburg **5** (1927), 7–23.

[28] ———, *Knotentheorie*, Springer, Berlin, 1932.

[29] Martin Scharlemann and Ruifeng Qiu, *A proof of the Gor don conjecture*, Advances in Mathematics **22** (2009), 2085–2106.

[30] Martin Scharlemann and Abigail Thompson, *Link genus and the Conway moves*, Comment. Math. Helv. **4** (1989), 527–535.