Unknotting genus one knots

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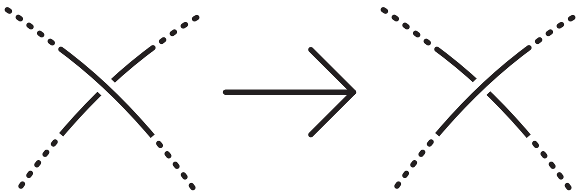
Joint work with Marc Lackenby

Theorem 1: [C, Lackenby] If K is a genus one knot with unknotting number one, then, up to equivalence, there is precisely one crossing change that unknots K, except when K is the figure-eight knot in which case there are precisely two unknotting crossing changes, up to equivalence.

Let K be a knot in S^3. The unknotting number of K is the minimum number of crossing changes required to turn K into the unknot.



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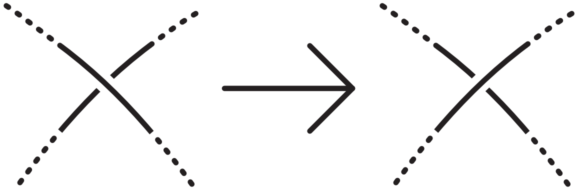
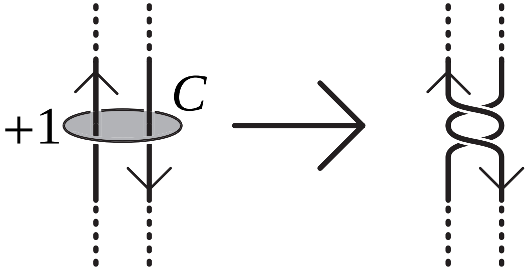


Let C be a simple closed curve in the complement of K that bounds an embedded disc which K intersects transversely in two points of opposite sign.

Then C is said to be a crossing circle for K and a crossing change on K is achieved by performing ±1 Dehn surgery along C.



A crossing change is unknotting if the resulting knot is the unknot.

Two crossing changes are equivalent if the surgery coefficients are the same and there is an ambient isotopy, keeping K fixed throughout, that takes one crossing circle to the other.

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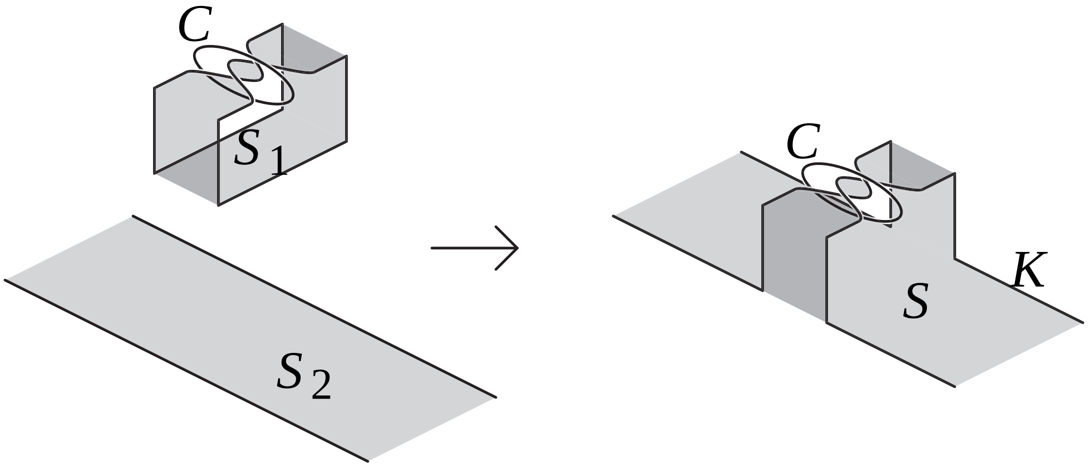
Theorem: [Osoinach] There are closed orientable 3-manifolds which may be obtained by performing surgery on infinitely many knots in S^3.

Theorem: [Scharlemann and Thompson, 1989] Let C be a crossing circle for a non-trivial knot K such that performing a crossing change along C unknots K. Then K has a minimal genus Seifert surface, S, which is obtained by plumbing two surfaces S_1 and S_2, where S_1 is a Hopf band. Moreover, there is an ambient isotopy, keeping K fixed throughout, that takes C to the associated crossing circle for S_1.



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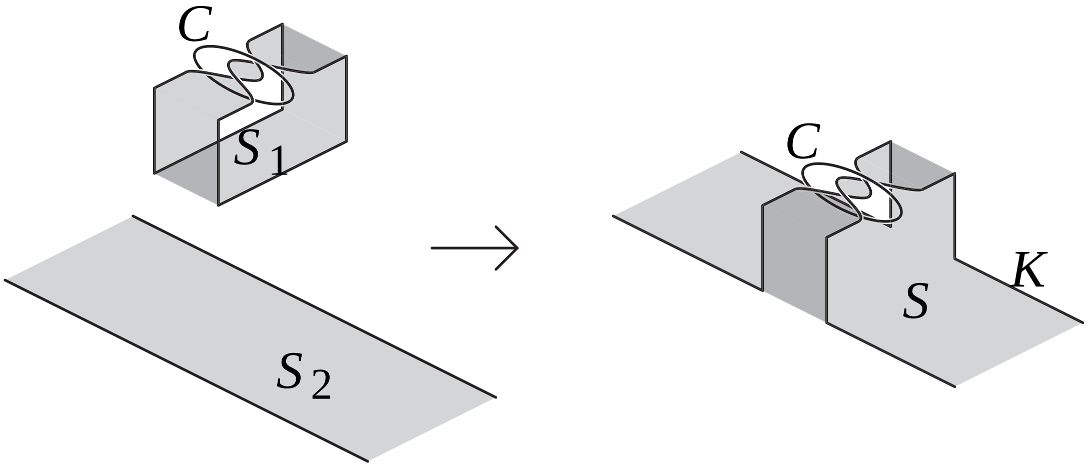

Corollary: A genus one knot has unknotting number one if and only if it is a doubled knot.

If K is hyperbolic, then it has only finitely many minimal genus Seifert surfaces, up to ambient isotopy keeping K fixed throughout, and there is an algorithm to find all of these surfaces.

So to detect if a knot has unknotting number one, we would like to inspect each minimal genus Seifert surface of K to see if it has a plumbed on Hopf band corresponding to an unknotting crossing change.

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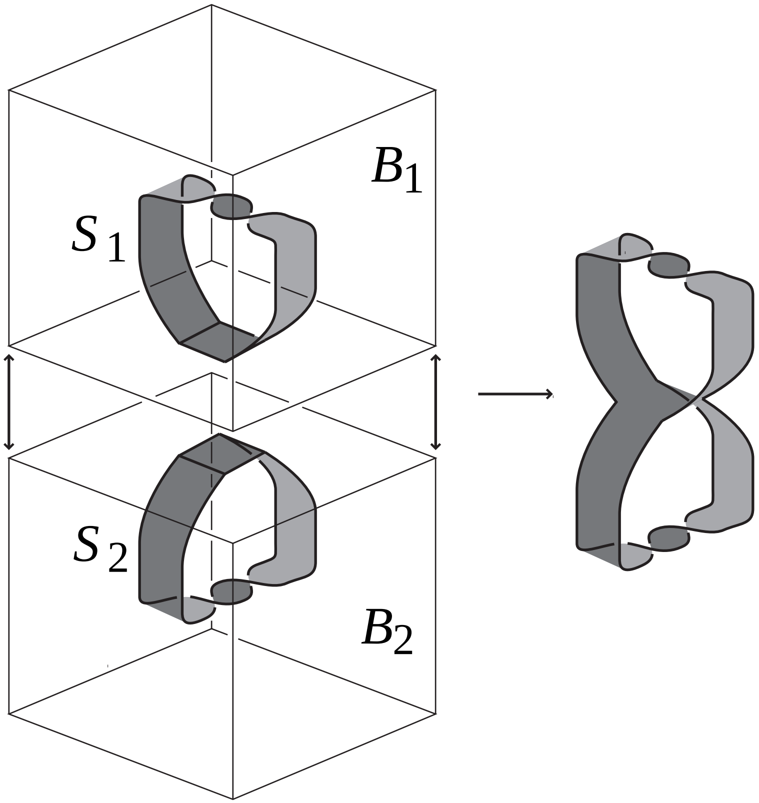
Question 2: How can one characterize decompositions of S as the pluming of two surfaces, the first of which is a Hopf band?

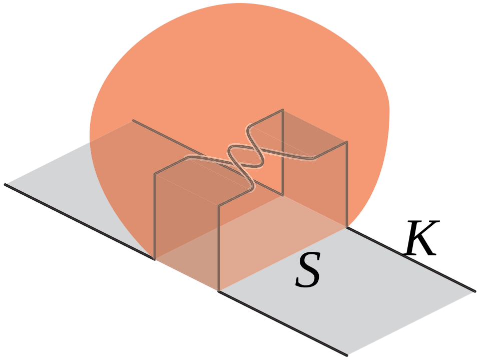
Suppose that S_1 and S_2 are compact orientable surfaces embedded in 3-balls B_1 and B_2. Suppose that the intersection of each S_i with ∂B_i is a square $I \times I$ such that $(I \times I) \cup \partial S_1 = I \times \partial I$ and $(I \times I) \cup \partial S_2 = \partial I \times I$. Then the surface in S^3 obtained by plumbing S_1 and S_2 is constructed by gluing the boundaries of B_1 and B_2 so that the two copies of $I \times I$ are identified in a way that preserves their product structures.



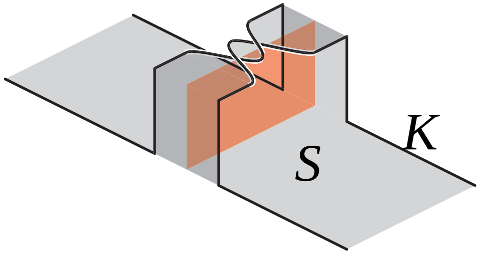
Suppose that a surface S can be obtained by plumbing in two ways, by combining $S_1 \subset B_1$ with $S_2 \subset B_2$, and by combining $S'_1 \subset B'_1$ with $S'_2 \subset B'_2$. We say that these are equivalent if there exists an ambient isotopy of the 3-sphere, leaving S invariant throughout, that takes S_i to S'_i $(i = 1, 2)$ and B_i to B'_i $(i = 1, 2)$.

Question 2: How can one characterize decompositions of S as the pluming of two surfaces, the first of which is a Hopf band?

Possible answer: With the 2-sphere that specifies the decomposition.



Better answer: With the following disc:



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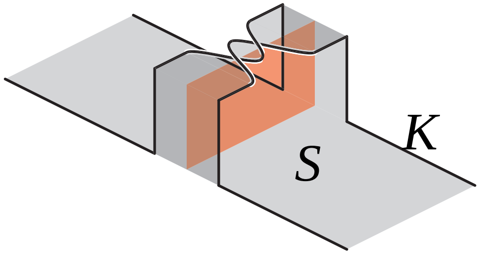

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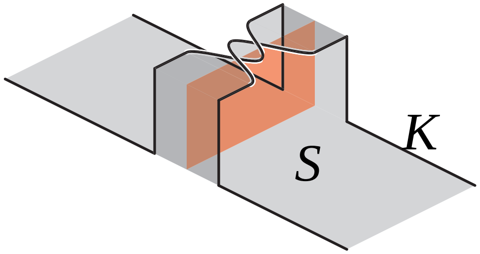

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Let S be a Seifert surface for a knot K. Let $N(S)$ be a small product neighbourhood $S \times I$. Let S_- and S_+ denote the two components of $S \times \partial I$. Let $M = S^3 - \mathrm{int}(N(S))$.

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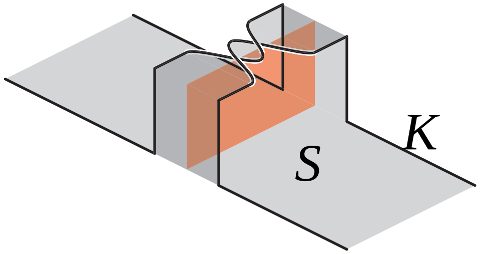

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A product disc is a disc properly embedded in M that intersects $\partial S \times I$ in two vertical arcs. It therefore intersects S_- in an arc and S_+ in an arc.

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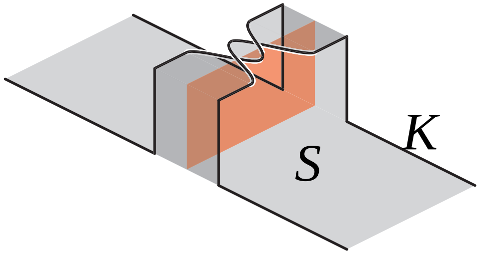

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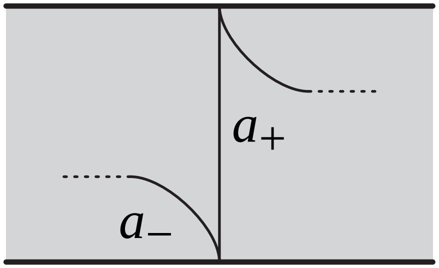
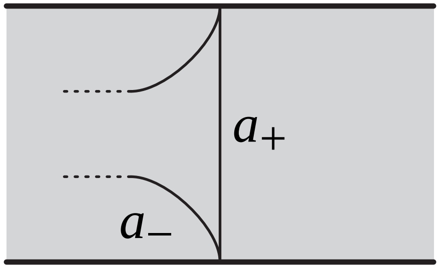
Let α_+ and α_- denote the projections of $S_+ \cap D$ and $S_- \cap D$ to S.

A product disc D is clean if α_+ and α_- have disjoint interiors, up to ambient isotopy of M that fixes $\partial S \times I$ throughout.



Alternating



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Theorem: [C, Lackenby] Let S be a Seifert surface for a knot K. Then there is a one-one correspondence between the following:

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2. Clean alternating product discs for S, up to ambient isotopy that leaves $N(S)$ invariant and maintains the disc as a product disc throughout.

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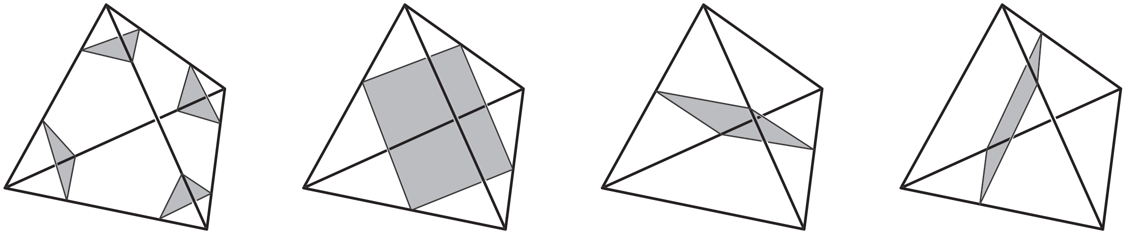
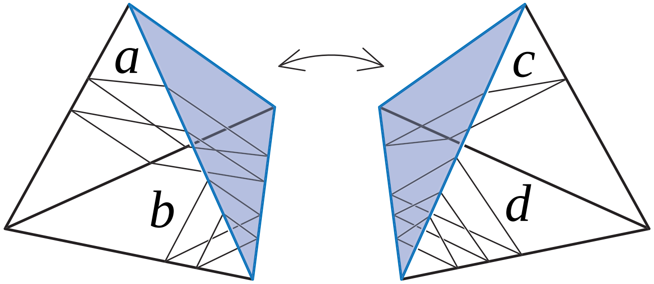
Standard results from normal surface theory imply that alternating product discs can be isotoped to be normal relative to any triangulation of M.

Question: Why does this not lead us quickly to an algorithm to search for clean alternating product discs?

Definition: A normal surface in a triangulated 3-manifold is a properly embedded surface which intersects each tetrahedron of the triangulation in a collection of disjoint triangles and quadrilaterals.



How does one search for normal surfaces?



$$a + b = c + d$$

Let $x_1, \dots, x_n$ be the number of discs of each type in F, a normal surface.

$n = 7 \times$ (number of tetrahedra in M).

Then $x_1, \dots, x_n$ satisfy a system of linear equations of the form:

$$x_h + x_j = x_k + x_l$$
$$x_i \geq 0, 1 \leq i \leq n.$$

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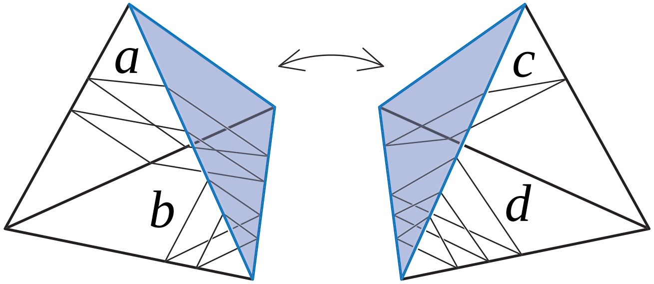

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So F corresponds to an element of $(\mathbb{N} \cup \{0\})^n$ which we shall call $g(F)$. $g(F)$ is a solution to the matching equations.

The matching equations for M have infinitely many solutions in general.

But, the matching equations will always have a finite list of solutions, called fundamental solutions, which give rise to all solutions as linear sums.

The fundamental solutions may be found algorithmically.

Not all the fundamental solutions necessarily correspond to normal surfaces, but those which do correspond to a set of normal surfaces, $N_1, \ldots, N_m$, called fundamental surfaces.

Does M contain a clean, alternating product disc, D, corresponding to an unknotting crossing change?

If so then

$$g(D) = \sum_{j=1}^{m} a_j g(N_j)$$

where $a_j \geq 0$ for all j.

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Fact 1: The addition of vectors corresponding to normal surfaces is Euler characteristic additive, in the sense that

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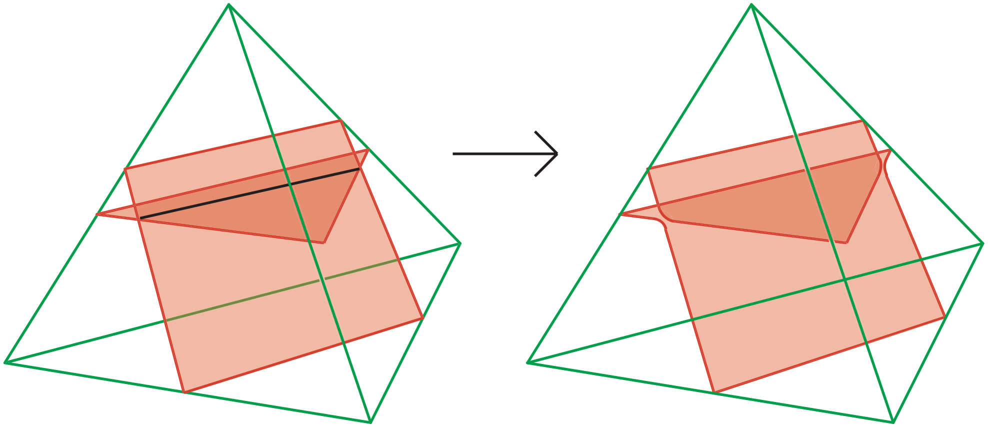
Fact 2: If F is a normal surface and $g(F) = g(F_1) + g(F_2)$ for normal surfaces F_1 and F_2, then F is obtained from the disjoint union of F_1 and F_2 by performing regular switches:



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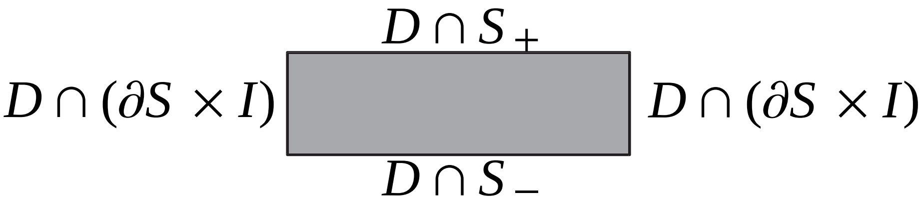
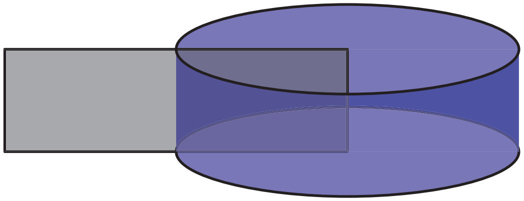
Answer: Vertical annuli.





These are annuli in M with one boundary component on each of S_+ and S_-. It is not clear how to bound their provenance in expressions of clean alternating product discs as normal sums of fundamental surfaces.

Proposition: Let K be a fibred knot with fibre S. Let $h : S \to S$ be the monodromy, where $h|\partial S$ is the identity. Then, there is a one-one correspondence between the following:

1. Clean essential product discs for S, up to ambient isotopy that leaves $N(S)$ invariant and maintains the disc as a product disc throughout.

2. Properly embedded essential arcs α in S, up to isotopy of α in S, such that $h(\alpha)$ and α can be ambient isotoped, keeping their boundaries fixed, so that their interiors are disjoint.

Moreover, the product disc is alternating if and only if α and $h(\alpha)$ are alternating.

Corollary: Let K be a hyperbolic fibred knot with fibre S. Suppose that S is obtained by plumbing two surfaces, the first of which is a Hopf band. Then it does so in infinitely many inequivalent ways.

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Proof: Let α be the arc corresponding to the hypothesized decomposition of S.

Since K is hyperbolic, h is pseudo-Anosov. Hence $h^n(\alpha)$ are all distinct up to isotopy.

These arcs correspond to infinitely many distinct ways to decompose S as the plumbing of two surfaces, the first of which is a Hopf band.

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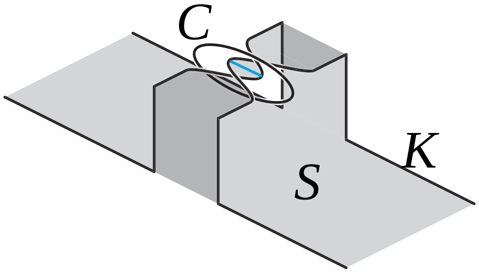
However, all of these plumbings correspond to equivalent surgery curves.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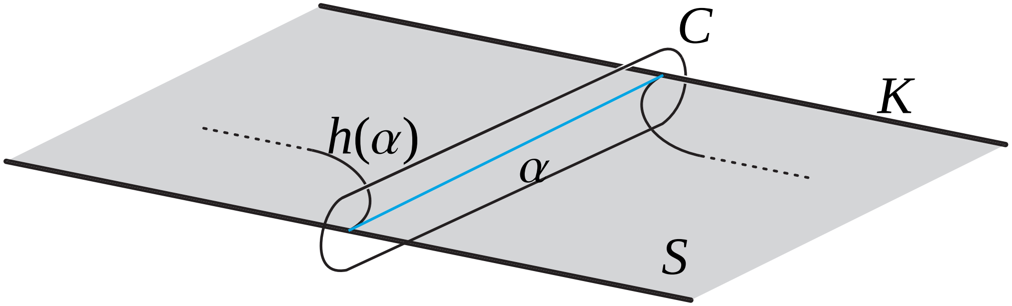
S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We say that two arcs α and β on S are h-equivalent if $h^n(\alpha)$ is ambient isotopic to β for some integer n.

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Unknotting crossing circles for fibred knots correspond to certain monodromy-equivalence classes of arcs on a fibre.

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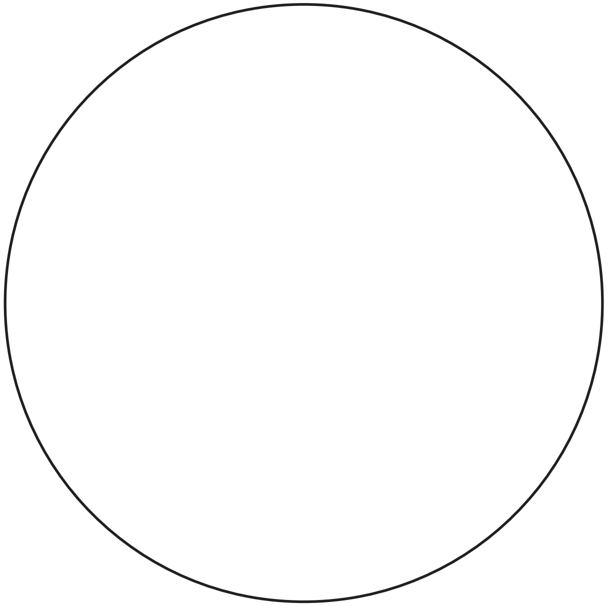

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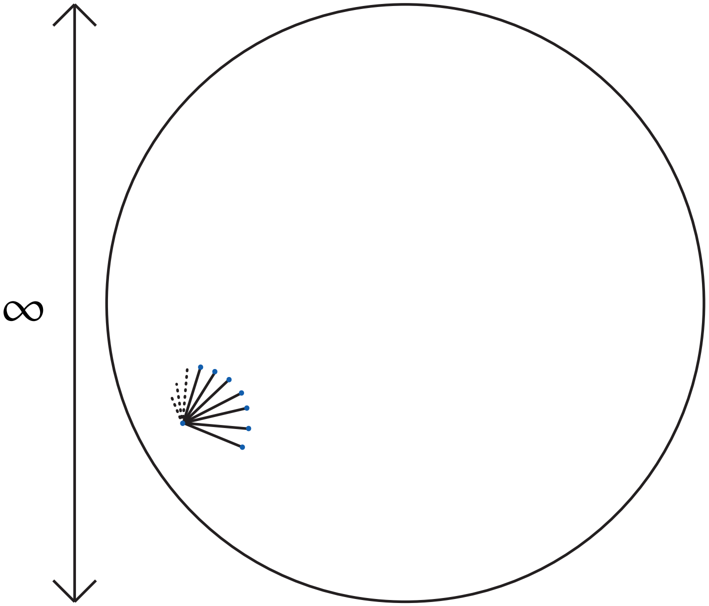

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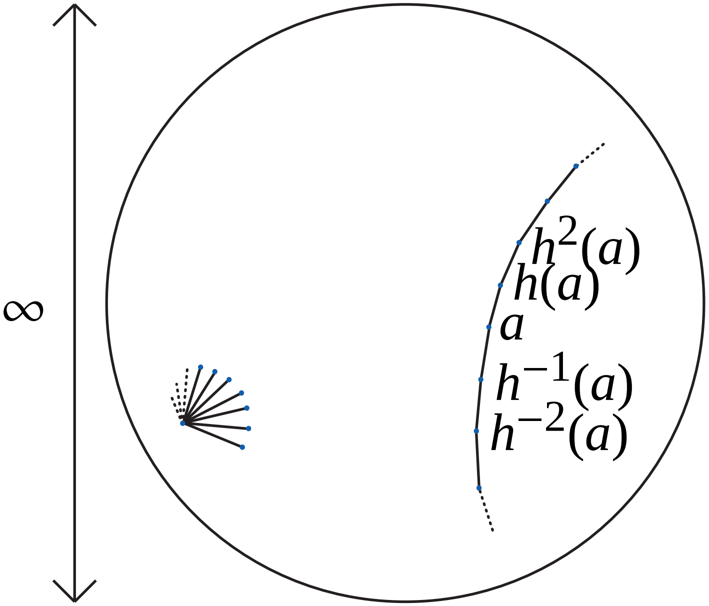

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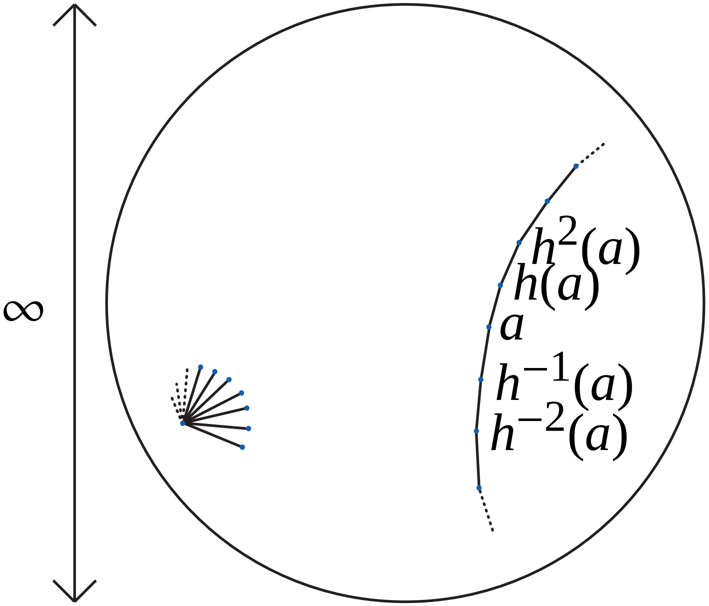

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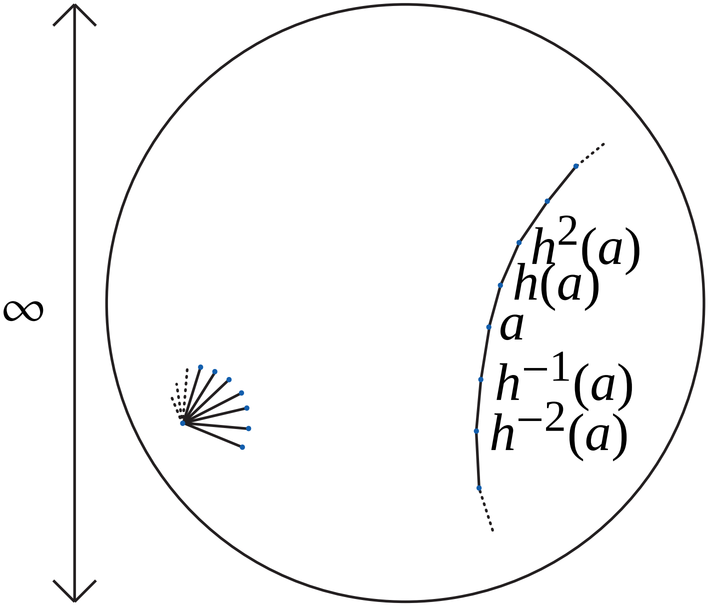

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Theorem: [Schleimer, Mazur] $\mathcal{A}(S)$ is δ-hyperbolic.

Let $\gamma \subseteq S$ be an essential arc corresponding to a vertex of $\mathcal{A}(S)$. Let $d(\gamma, h(\gamma)) \leq D$. Then $\exists R_{D,\delta} > 0$ such that every monodromy-equivalence class of arcs corresponding to an unknotting crossing change intersects $B_{R_{D,\delta}}(\delta)$.

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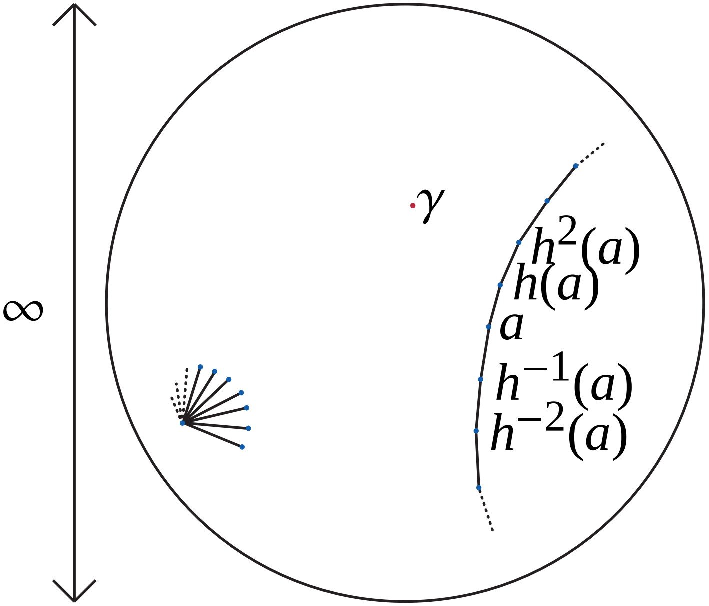

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Let $\gamma \subseteq S$ be an essential arc corresponding to a vertex of $\mathcal{A}(S)$. Let $d(\gamma, h(\gamma)) \leq D$. Then $\exists R_{D,\delta} > 0$ such that every monodromy-equivalence class of arcs corresponding to an unknotting crossing change intersects $B_{R_{D,\delta}}(\delta)$.

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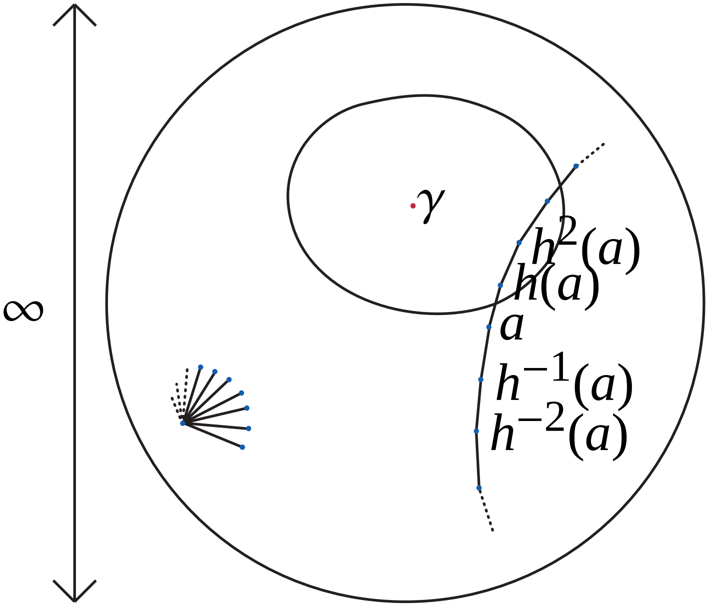

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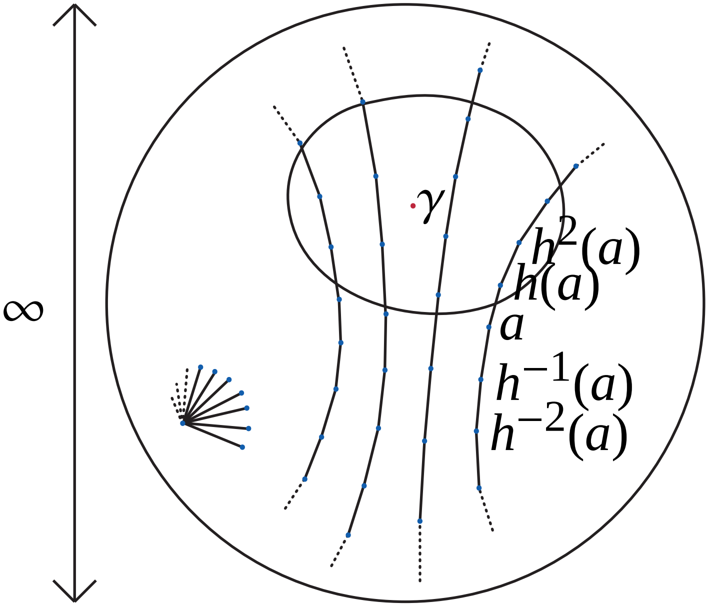

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Theorem 1: **[C, Lackenby]** If K is a genus one knot with unknotting number one, then, up to equivalence, there is precisely one crossing change that unknots K, except when K is the figure-eight knot in which case there are precisely two unknotting crossing changes, up to equivalence.

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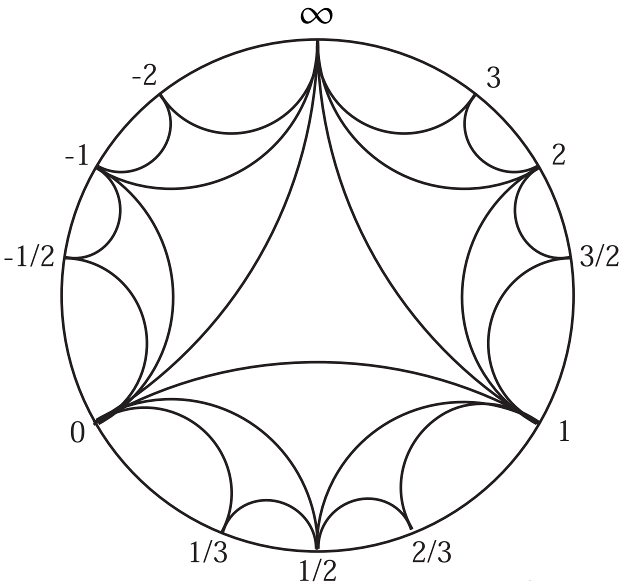
This completes the proof of Theorem 1 in the non-fibred case.

When K is fibred, with fibre S, consider the arc complex, $\mathcal{A}(S)$, of S.



The 1-skeleton of this is the Farey graph. It has vertices for each element of $\mathbb{Q} \cup \infty$. Two slopes $\frac{p}{q}$ and $\frac{p'}{q'}$ are joined by an arc iff $|pq' - qp'| = 1$. If one removes the vertices of $\mathcal{A}(S)$ then it identifies naturally with $\mathbb{H}^2$.

Let $h : S \to S$ be an orientation preserving homeomorphism of S. This h induces an automorphism of $H_1(S)$, and hence an element of $SL(2, \mathbb{Z})$. The image of this in $PSL(2, \mathbb{Z})$ corresponds to a Möbius map, which induces an orientation-preserving isometry of $\mathbb{H}^2$.

This will be elliptic, parabolic or loxodromic depending on whether h is periodic, reducible (with infinite order) or pseudo-Anosov.

Theorem: [C, Lackenby] Let $h\colon S \to S$ be an orientation-preserving homeomorphism of the genus one surface with one boundary component, which is not isotopic to the identity. Then, up to h-equivalence, there are at most two essential properly embedded arcs α in S such that α and $h(\alpha)$ can be isotoped to be disjoint. Moreover, if h is periodic, then there is at most one such arc up to h-equivalence.

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Then h leaves an essential arc invariant up to isotopy. Hence the element of $\mathrm{SL}(2,\mathbb{Z})$ is conjugate to

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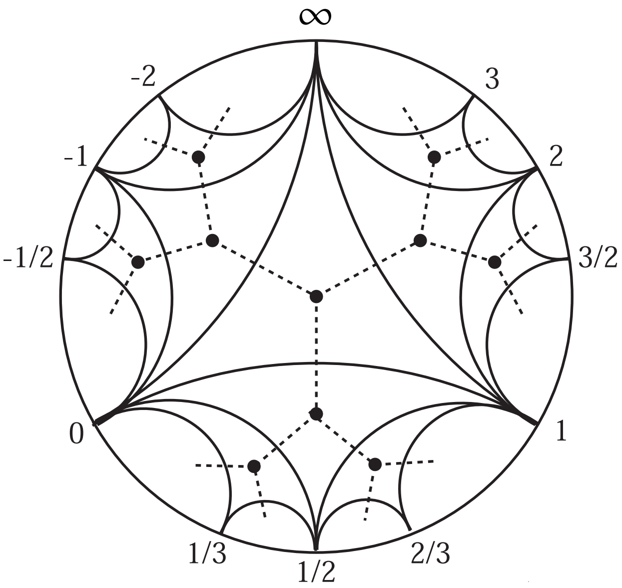
Now, up to h-equivalence, the only properly embedded essential arcs α in S such that $h(\alpha)$ and α can be made disjoint are ∞ and 0, and that the latter only arises if $n = \pm 1$. Hence the theorem holds in this case.

In the cases where h is periodic or pseudo-Anosov, consider the dual tree, T, of $\mathcal{A}(S)$.



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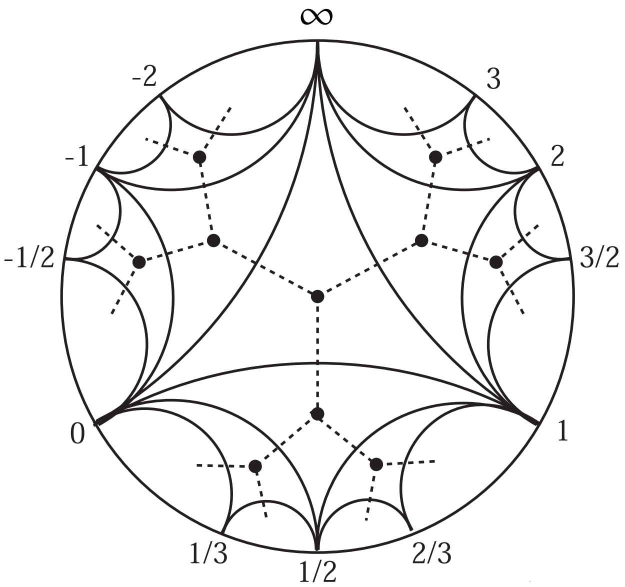

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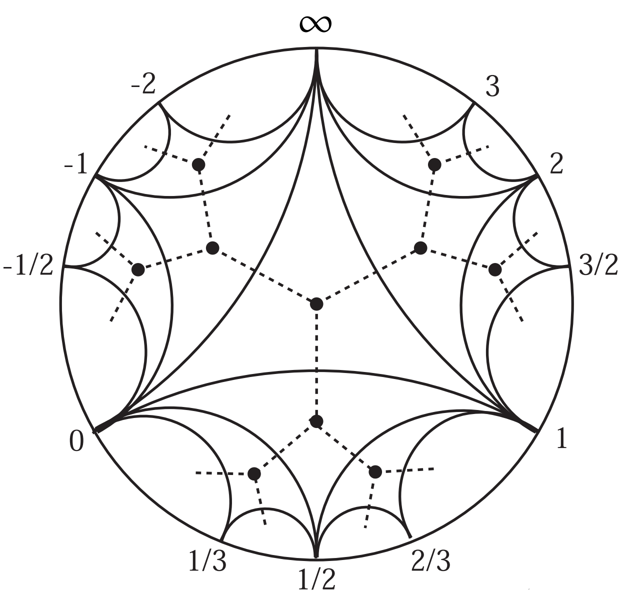

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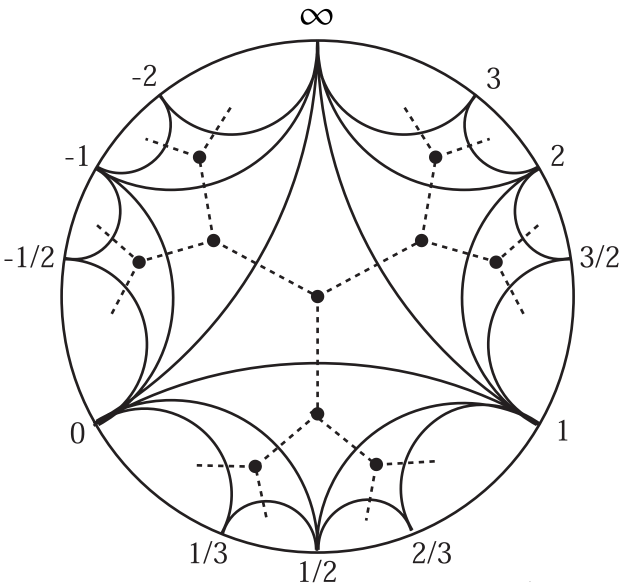

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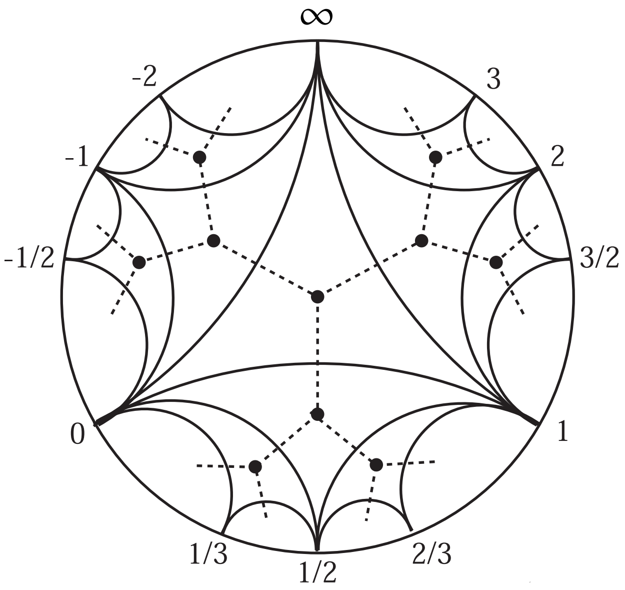

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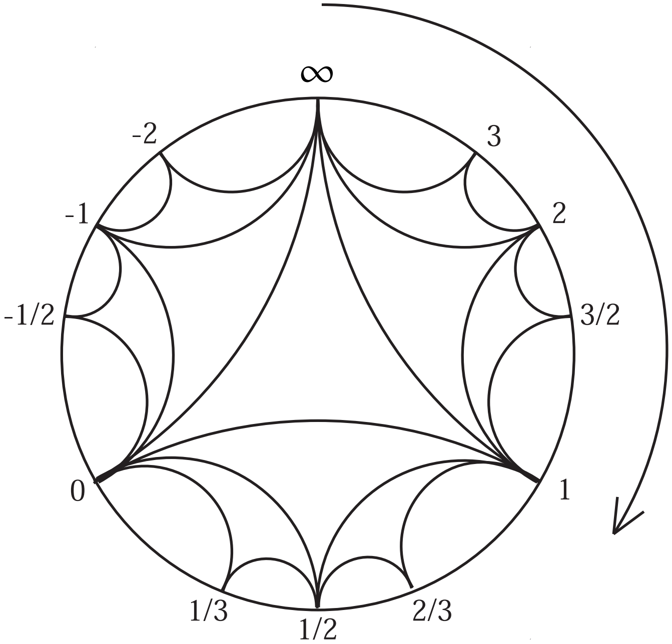
2) are fixed point free but preserve a unique invariant line.

In the case of the trefoil the monodromy induces an isometry conjugate to a rotation by $\frac{2\pi}{3}$.



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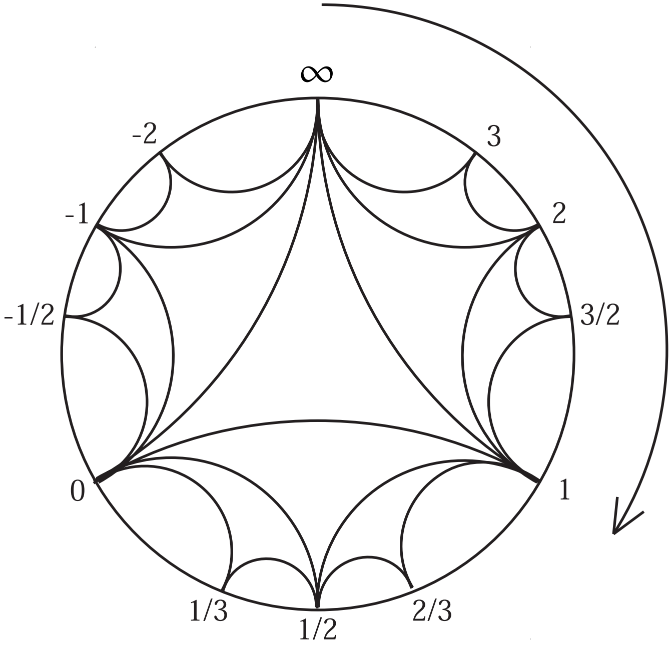
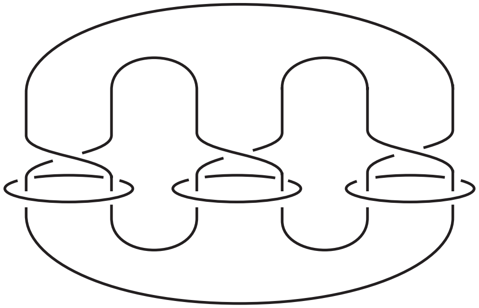
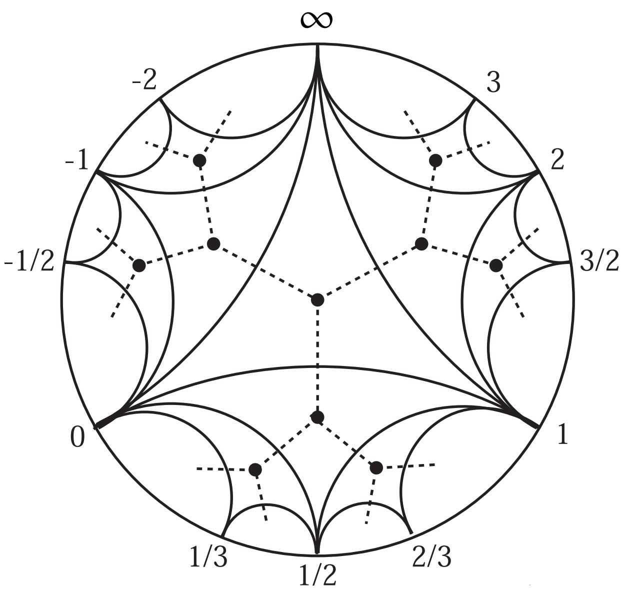




In the case of the figure-eight knot the monodromy induces a loxodromic isometry of $\mathbb{H}^2$ which fixes an axis of the dual tree.



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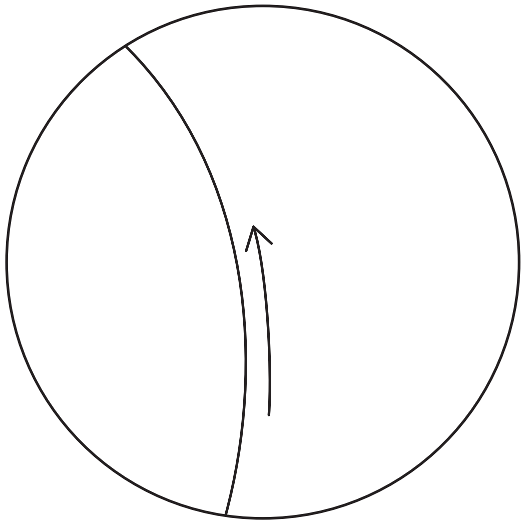

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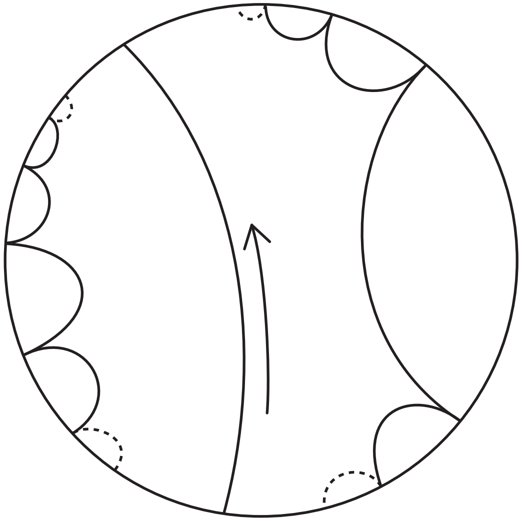

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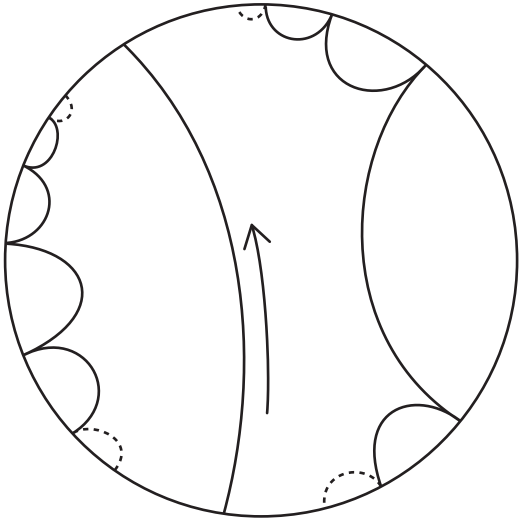
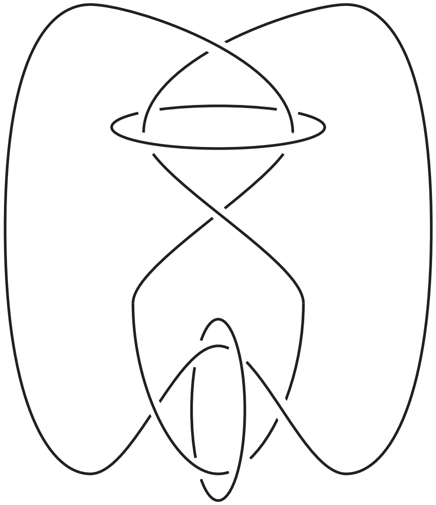


So, there are at most two ways of unknotting the figure-eight knot.

And here they are:



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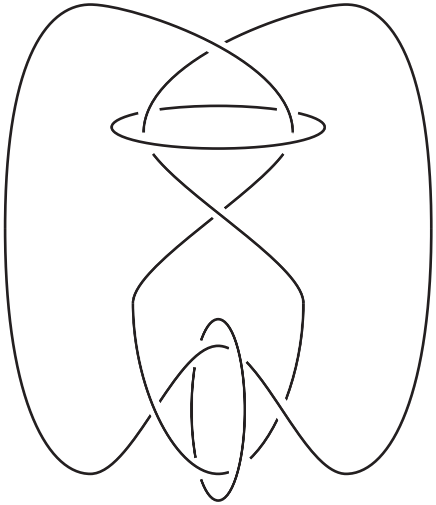

THE END