Equations of planes and lines in space: typical problems.

A problem similar to (one of) the problems below will be offered in Midterm 2 and Final.

1. Find an equation of the plane passing through the point $(1, 2, -3)$ and perpendicular to the vector $\langle 3, 2, 1\rangle$.

Solution. Any plane perpendicular to the vector $\langle 3, 2, 1\rangle$ has an equation $3x + 2y + z + D = 0$. Plug the point to find D: $3 \cdot 1 + 2 \cdot 2 + 1 \cdot (-3) + D = 0$, $D = -4$. *Answer*: $3x + 2y + z - 4 = 0$.

2. Find an equation of the plane passing through the point $(2, 1, 2)$ and parallel to the plane $x + 3y - z + 5 = 0$.

Solution. Any plane parallel to this plane has an equation $x + 3y - z + D = 0$. Plug the point to find D: $2 + 3 \cdot 1 - 2 + D = 0$, $D = -3$. *Answer*: $x + 3y - z - 3 = 0$.

3. Find an equation of the plane passing through the points $P = (1, 1, 2)$, $Q = (2, 2, 0)$, $R = (1, 3, 1)$.

Solution. $\overrightarrow{PQ} = \langle 1, 1, -2\rangle$, $\overrightarrow{PR} = \langle 0, -2, 1\rangle$,

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 0 & 2 & -1 \end{vmatrix} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}.$$

Our plane is perpendicular to the last vector, hence it has an equation $3x + y + 2z + D = 0$. Plug P (or Q, or R) to find D: $3 + 1 + 4 + D = 0$, $D = -8$. *Answer*: $3x + y + 2z - 8 = 0$.

4. Find an equation of the plane passing through the points $P = (0, 1, 2)$, $Q = (2, 2, 1)$ and perpendicular to the plane $2x - y - 2z + 7 = 0$.

Solution. A vector perpendicular to the plane we need is perpendicular to the vector $\overrightarrow{PQ} = \langle 2, 1, -1\rangle$ and to the vector $\langle 2, -1, -2\rangle$ perpendicular to the given plane. Hence, the vector

$$\langle 2, 1, -1\rangle \times \langle 2, -1, -2\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 2 & -1 & -2 \end{vmatrix} = -3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$$

is perpendicular to our plane. The equation we need is $-3x + 2y - 4z + d = 0$. Plug P (or Q) to find D: $2 \cdot 1 - 4 \cdot 2 + d = 0$, $D = 6$. *Answer*: $-3x + 2y - 4z + 6 = 0$.

5. Find the distance from the point $(2, 1, 1)$ to the plane $2x - 2y + z + 2 = 0$.

Solution.

$$\text{the distance} = \frac{|2 \cdot 2 - 2 \cdot 1 + 1 - 2|}{\sqrt{2^2 + 2^2 + 1^2}} = \frac{5}{3}.$$

6. Prove that the planes $2x + 2y - 2z + 3 = 0$ and $3x + 3y - 3z + 2 = 0$ are parallel and find the distance between them.

Solution. Planes are parallel, if they have equations which are identical with the exception of the constant terms. Our equations are not like this, but become like this after a multiplication by the appropriate factors:

$$3(2x+2y-2z+3) = 6x+6y-6z+9 = 0,$$
$$2(3x+3y-3z+2) = 6x+6y-6z+4 = 0;$$
$$\text{the distance} = \frac{|9-4|}{\sqrt{6^2+6^2+6^2}} = \frac{5}{6\sqrt{3}}.$$

7. Find the parametric equation of a line passing through the points $P = (2,3,1)$ and $Q = (3,2,3)$.

Solution. The vector $\overrightarrow{PQ} = \langle 1,-1,2\rangle$ is parallel to the line. Thus the parametric equations are

$$\begin{aligned} x &= 2+t \\ y &= 3-t \\ z &= 1+2t \end{aligned} \qquad \text{or} \qquad \begin{aligned} x &= 3+t \\ y &= 2-t \\ z &= 3+2t \end{aligned}$$

8. Find the intersection point of the plane $2x-5y+z+2=0$ with the line $x = 5-2t,\ y=-1+t,\ z=7-3t$.

Solution. Plug the expressions for x, y, z from the equation of the line to the equation of the plane. We get: $2(5-2t)-5(-1+t)+(7-3t)+2 = 24-12t = 0$. Hence, $t = 2$ and $x = 5-2\cdot 2 = 1, y = -1+2 = 1, z = 7-3\cdot 2 = 1$. *Answer*: $(1,1,1)$.

9. Find the parametric equations of the line passing through the point $(1,3,2)$ and perpendicular to the plane $x+y-2z+7=0$.

Answer:

$$\begin{aligned} x &= 1+t \\ y &= 3+t \\ z &= -2-2t \end{aligned}$$

10. Find the parametric equations of the line of intersection of the planes $x-y+z+2 = 0$ and $2x+y-z+1=0$.

Solution. First, find the coordinates of a point on our line. For this, we need a solution of the system of equations,

$$\begin{cases} x-y+z=0 \\ 2x+y-z+1=0 \end{cases}$$

This system has infinitely many solutions, to specify one, we assign a value to one of the unknowns, say, take $z = 0$. The remaining system has a (unique) solution, $x=-1,\ y=1$. Thus the point $(-1,1,0)$ lies on our line.

Next, our line should be perpendicular to the vector $\langle 1,-1,1\rangle$ perpendicular to the first plane, and to the vector $\langle 2,1,-1\rangle$ perpendicular to the second plane. Hence, our line is parallel to the cross-product of these two vectors,

$$\langle 1,-1,1\rangle \times \langle 2,1,-1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = 0\mathbf{i}+3\mathbf{j}+3\mathbf{k} = \langle 0,3,3\rangle$$

(we may cancel this by 3: $\langle 0, 1, 1\rangle$). The equation of the line is

$$\begin{aligned} x &= -1 \\ y &= \quad 1 + t \\ z &= \qquad t \end{aligned}$$

11. Find the distance from the point $P = (1, 2, 1)$ to the line

$$\begin{aligned} x &= \quad 1 + t \\ y &= -2 + t \\ z &= \quad 2 - t \end{aligned}$$

Solution. The line given passes through the point $Q = (1, -2, 2)$ and is parallel to the vector $\mathbf{v} = \langle 1, 1, -1\rangle$. Let $\mathbf{u} = \overrightarrow{PQ} = \langle 0, -4, 1\rangle$. The formula says that

$$\text{the distance} = \frac{|\mathbf{u} \times \mathbf{v}|}{|\mathbf{v}|}.$$

Since

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 4 & -1 \\ 1 & 1 & -1 \end{vmatrix} = 3\mathbf{i} + \mathbf{j} + 4\mathbf{k} = \langle 3, 1, 4\rangle,$$

$$\text{the distance} = \frac{\sqrt{3^2 + 1^2 + 4^2}}{\sqrt{1^2 + 1^2 + 1^2}} = \sqrt{\frac{26}{3}}.$$