Math 21C

Chapter 14 Problems

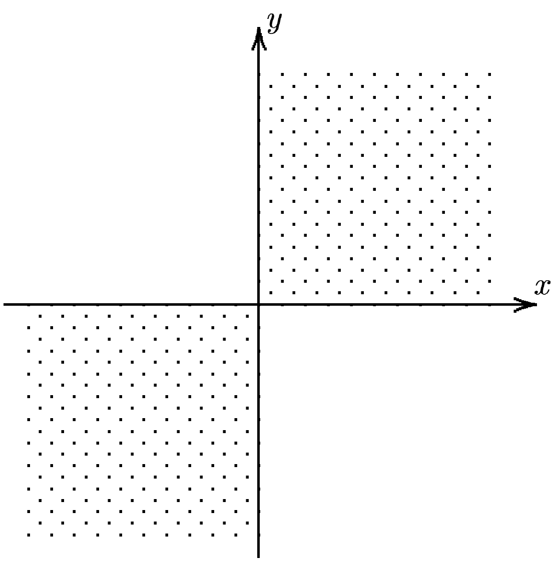
1. Describe and sketch the domain of each of the following functions.

(a) $f(x, y) = \sqrt{xy}$.

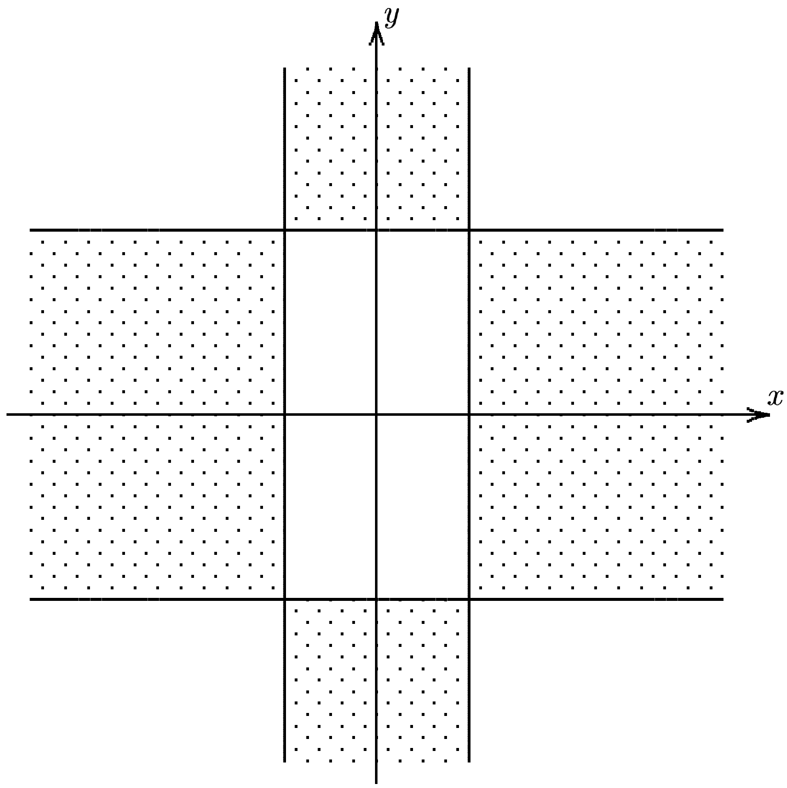
Solution. The domain, $xy \geq 0$, is the union of I and III quadrant (including the coordinate axes). See Figure (a).

(b) $f(x, y) = \sqrt{(1 - x^2)(y^2 - 4)}$.

Solution. The domain is described by the inequalities $|x| \leq 1,\ |y| \geq 2$, or $|x| \geq 1, |y| \leq 2$. See Figure (b).



(a)



(b)

2. Find the limit or show that it does not exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x - y}{x + y}$.

Solution. If $x \neq 0, y = 0$, then $\dfrac{x - y}{x + y} = 1$; if $y \neq 0, x = 0$, then $\dfrac{x - y}{x + y} = -1$. Thus, the limit along the y axis is 1, while the limit along the x axis is -1. Therefore, our limit does not exist.

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x + 1)^2 - (y + 1)^2}{(x - 1)^2 - (y - 1)^2}$.

Solution. Since

$$\frac{(x + 1)^2 - (y + 1)^2}{(x - 1)^2 - (y - 1)^2} = \frac{(x - y)(x + y + 2)}{(x - y)(x + y - 2)} = \frac{x + y + 2}{x + y - 2},$$

the limit is $\lim_{(x,y)\to(0,0)} \frac{x+y+2}{x+y-2} = -1.$

3. Find the parametric equations of the tangent line to the curve

$$x = t^2 - 1, y = t^3, z = t^4$$

at the point $(0, -1, 1)$.

Solution. Since at our point $y = t^3 = -1$, we see that $t = -1$. Check that our point corresponds to $t = -1$: $x = (-1)^2 - 1 = 0$, $z = (-1)^4 = 1$. The components of the vector paralell to the tangent line are $x'(-1) = 2 \cdot (-1) = -2$, $y'(-1) = 3 \cdot (-1)^2 = 3$, $z'(-1) = 4 \cdot (-1)^3 = -4$. Thus, the tangent line has parametric equations $x = 0 - 2t$, $y = -1 + 3t$, $z = 1 - 4t$. smallskip

4. Find partial derivatives $\frac{\partial z}{\partial u}$, $\frac{\partial z}{\partial v}$ for given functions $z = z(x, y)$ using the given expressions of x and y as functions of u and v. Express the derivatives as functions of u and v.

(a) $z = \sin(xy)$, $x = ue^v, y = ue^{-v}$.

Solution.

$$z_u = y\cos(xy) \cdot e^v + x\cos(xy) \cdot e^{-v} = ue^{-v}\cos(u^2)e^v + ue^v\cos(u^2)e^{-v} = 2u\cos(u^2)$$
$$z_v = y\cos(xy) \cdot ue^v - x\cos(xy) \cdot ue^{-v} = ue^{-v}\cos(u^2)ue^v - ue^v\cos(u^2)ue^{-v} = 0$$

(b) $z = \ln\left|\frac{x-y}{x+y}\right|$, $x = u^2 + v^2, y = 2uv$.

Solution.

$$\begin{aligned}
z_x &= \frac{x+y}{x-y} \cdot \frac{(x+y)-(x-y)}{(x+y)^2} = \frac{2y}{x^2-y^2},\\
z_y &= \frac{x+y}{x-y} \cdot \frac{-(x+y)-(x-y)}{(x+y)^2} = -\frac{2x}{x^2-y^2}\\
z_u &= 2uz_x + 2vz_y = \frac{4uy-4vx}{x^2-y^2} = \frac{8u^2v - 4u^2v - 4v^3}{(u^2+v^2)^2 - 4u^2v^2} = \frac{4v(u^2-v^2)}{(u^2-v^2)^2} = \frac{4v}{u^2-v^2}\\
z_v &= 2vz_x + 2uz_y = \frac{4vy-4ux}{x^2-y^2} = \frac{8uv^2 - 4u^3 - 4uv^2}{(u^2+v^2)^2 - 4u^2v^2} = \frac{4u(v^2-u^2)}{(u^2-v^2)^2} = -\frac{4u}{u^2-v^2}
\end{aligned}$$

5. Find the direction of the fastest growth of the function

(a) $x^2 + 2y^2 + 3z^2$ at the point $(6, 3, 1)$

(b) $e^{x+y}\cos(x-y)$ at the point $(0, 0)$

and find the directional derivative of the given function at the given point in the direction of the fastest growth.

Solution. (a) The gradient of the given function is $\langle 2x, 4y, 6z\rangle$. At the given point it is $\langle 12, 12, 6\rangle$. The length of the gradient is $\sqrt{12^2 + 12^2 + 6^2} = 18$. The direction of the fastest growth, which is the direction of the gradient is

$$\frac{\langle 12, 12, 6\rangle}{18} = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle;$$

the directional derivative in this direction is equal to the length of the gradient, that is, to 18.

(b) The gradient of the given function is $\langle e^{x+y}(\cos(x-y)-\sin(x-y)), e^{x+y}(\cos(x-y)+\sin(x-y))\rangle$. At the given point it is $\langle 1, 1\rangle$. The length of the gradient is $\sqrt{2}$. The direction of the fastest growth, which is the direction of the gradient is $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$; the directional derivative in this direction is equal to the length of the gradient, that is, to $\sqrt{2}$.

6. Find the equation of the tangent plane to the surface $z = x^2 - y^2$ at the point $(3, 2, 5)$.

Solution. The surface is $f(x, y, z) = 0$ where $f(x, y, z) = x^2 - y^2 - z$. The gradient of f at the point $(3, 2, 5)$ is $\langle 6, -4, -1\rangle$. Thus, the equation of the tangent plane is $6x - 4y - z + D = 0$ where D can be determined from the condition that the point $(3, 2, 5)$ lies in this plane. Thus, $D = -5$, and the answer is $6x - 4y - z - 5 = 0$.

7. Find all local maxima, local minima and saddle points for the function

$$f(x, y) = x^2 + xy + y^2 + \frac{1}{x} + \frac{1}{y}, \; x > 0, y > 0.$$

Solution.

$$\frac{\partial f}{\partial x} = 2x + y - \frac{1}{x^2}, \; \frac{\partial f}{\partial y} = x + 2y - \frac{1}{x^2};$$

$$\frac{\partial^2 f}{\partial x^2} = 2 + \frac{2}{x^3}, \; \frac{\partial^2 f}{\partial x \partial y} = 1, \; \frac{\partial^2 f}{\partial y^2} = 2 + \frac{2}{y^3}$$

Find zeroes of the first derivatives:

$$\begin{array}{l} 2x + y = \frac{1}{x^2} \\ x + 2y = \frac{1}{y^2} \end{array} \quad \Rightarrow \quad \begin{array}{l} 2x^2 + xy = \frac{1}{x} \\ xy + 2y^2 = \frac{1}{y} \end{array} \quad \Rightarrow \quad 2(x^2 - y^2) = \frac{1}{x} - \frac{1}{y} = \frac{y - x}{xy}.$$

This means that either $x = y$, or $2(x+y) = -\frac{1}{xy}$; the latter is impossible for $x > 0, y > 0$. So, $x = y$, $3x = \frac{1}{x^2}, x = \frac{1}{\sqrt[3]{3}}$. Thus, there is only one critical point, $\left(\frac{1}{\sqrt[3]{3}}, \frac{1}{\sqrt[3]{3}}\right)$. At this point, $\frac{\partial^2 f}{\partial x^2} = 8, \frac{\partial^2 f}{\partial x \partial y} = 1, \frac{\partial^2 f}{\partial y^2} = 8$, and, since $\begin{vmatrix} 8 & 1 \\ 1 & 8 \end{vmatrix} > 0$, $8 > 0$, this is a local minimum.

Answer. A local minimum at $\left(\frac{1}{\sqrt[3]{3}}, \frac{1}{\sqrt[3]{3}}\right)$.

8. Find all local maxima, local minima and saddle points for the function

$$f(x, y) = x^2 + xy + y^2 + \frac{1}{x} + \frac{1}{y}, \; x > 0, y > 0.$$

Solution.

$$\frac{\partial f}{\partial x} = 4x^3 - 2(x+y),\ \frac{\partial f}{\partial y} = 4y^3 - 2(x+y);$$

$$\frac{\partial^2 f}{\partial x^2} = 12x^2 - 2,\ \frac{\partial^2 f}{\partial x \partial y} = -2,\ \frac{\partial^2 f}{\partial y^2} = 12y^2 - 2$$

Find zeroes of the first derivatives:

$$\begin{matrix} 4x^3 = 2(x+y) \\ 4y^3 = 2(x+y) \end{matrix} \quad \Rightarrow \quad \begin{matrix} x^3 - y^3 = 0 \Rightarrow x = y \\ 4x^3 = 4x \Rightarrow x = 0,\ 1,\ \text{or} - 1 \end{matrix}$$

Thus, there are three critical points, $(-1,-1)$, $(0,0)$ $(1,1)$. For $(-1,-1)$ and $(1,1)$, the computation with second partial derivatives shows that both are minima, for $(0,0)$ the determinant of secomd partial deribvatives is $\begin{vmatrix} -2 & -2 \\ -2 & -2 \end{vmatrix} = 0$, so our theorem does not give any answer. (Remark, that $(0,0)$ is neither a maximum, nor a minimum for our function, since $f(0,0) = 0$ and for small x, $f(x.x) = 2x^4 - 4x^2 < 0, f(x,-x) = 2x^4 > 0$.)

Answer. Local minima at $(1,1)$ and $(-1,-1)$, and a critical point which is neither a local maximum, nor a local minimum, at $(0,0)$.

9. Find all local maxima, local minima and saddle points for the function

$$f(x,y) = x^2 + axy + y^2,\ a \neq \pm 2$$

(dependingly on a).

Solution.

$$\frac{\partial f}{\partial x} = 2x + ay,\ \frac{\partial f}{\partial y} = ax + 2y;$$

$$\frac{\partial^2 f}{\partial x^2} = 2,\ \frac{\partial^2 f}{\partial x \partial y} = a,\ \frac{\partial^2 f}{\partial y^2} = 2$$

Find zeroes of the first derivatives:

$$\begin{matrix} 2x + ay = 0 \\ ax + 2y = 0 \end{matrix} \quad \Rightarrow \quad x = y = 0$$

(since $a \neq \pm 2$). The determinant of second partial derivatives is $\begin{vmatrix} 2 & a \\ a & 2 \end{vmatrix} = 4 - a^2$. If $|a| > 2$, this is negative, and the point is a saddle point. If $|a| < 2$, this is positive, and the point is a local minimum.

Answer. There is only one critical point, $(0,0)$; it is a local minimum, if $|a| < 2$ and a saddle if $|a| > 2$.

10. Find the maximum and minimum values of the function $f(x,y) = x^2 + xy + y^2$ under the constraint $x^2 - xy + y^2 = 1$.

Solution. The equations are

$$2x - y = \lambda(2x + y)$$
$$-x + 2y = \lambda(2x + y),$$
$$x^2 - xy + y^2 = 1.$$

Adding the first two equations we get $x + y = 3\lambda(x + y)$, subtracting we get $3x - 3y = \lambda(x - y)$. This leaves us with three possibilities: either $x = y$ and $x = -y$, that is, $x = y = 0$; but this is inconsistent with the first equation; or $\lambda = \frac{1}{3}$ and $x = -y$, which makes the first equation into $3x^2 = 1$, that is, $(x, y)=\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ or $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$; or $\lambda = 3$ and $x = y$, which gives $x^2 = 1$, $(x, y) = (1, 1)$ or $(-1, -1)$. Thus, Lagrange's method gives the points $(1, 1)$, $(-1, -1)$, $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$, $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, and the values of the function at these points are $3, 3, \frac{1}{3}, \frac{1}{3}$.

Answer. $\frac{1}{3}$ and 3.

11. Find the maximum and minimum values of the function $f(x, y, z) = xyz$ on the sphere $x^2 + y^2 + z^2 = 3$.

Solution. The equations are

$$yz = \lambda \cdot 2x$$
$$xz = \lambda \cdot 2y$$
$$xy = \lambda \cdot 2z$$
$$x^2 + y^2 + z^2 = 3$$

Multiplying the first three eqation, respectively, by x, y, and z, we get $2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2 = xyz$, which leaves us with two possibilities: $\lambda = 0$ or $x^2 = y^2 = z^2$.

In the first case, $yz = xz = xy = 0$ which means that two of x, y, z should be 0. Since $x^2 + y^2 + z^2 = 3$, there are 6 points corresponding to this case: $(0, 0, \pm\sqrt{3}), (0, \pm\sqrt{3}, 0), (\pm\sqrt{3}, 0, 0)$; the value of xyz at each of these points is 0.

In the second case (since $x^2 + y^2 + z^2 = 3$), $x^2 = y^2 = z^2 = 1$ and $x = \pm 1, y = \pm 1, z = \pm 1$. There are 8 such points, $(\pm 1, \pm 1, \pm 1)$, and the value of xyz of each of these points is 1 or -1.

Thus, the maximum value of the function is 1 (taken at the points $(1, 1, 1), (1, -1, -1), (1, -1, 1), (-1, -1, 1)$) and the minimum value is -1 (taken at the points $(-1, -1, -1), (-1, 1, 1), (1, -1, 1), (1, 1, -1)$).