Problem 1. Prove that the series

$$\sum_{n=0}^{\infty} \frac{2^n + 3^n}{5^n}$$

converges and find its sum.

Solution. This is the sum of two geometric series, $\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n + \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n$; both converge, since $\frac{2}{5} < 1$, $\frac{3}{5} < 1$. The sum is

$$\frac{1}{1 - \frac{2}{5}} + \frac{1}{1 - \frac{3}{5}} = \frac{5}{3} + \frac{5}{2} = \frac{25}{6}.$$

Problem # 2. Does the series

$$\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 10}$$

converge or diverge? Justify your answer.

Solution. The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, and

$$\lim_{n\to\infty} \left[\frac{n^2}{n^3 + 10} \div \frac{1}{n} \right] = \lim_{n\to\infty} \frac{n^3}{n^3 + 10} = 1.$$

Hence, the given series diverges by the limit comparison test.

Problem # 3. Does the series

$$\sum_{n=1}^{\infty} \frac{n^n}{n! \cdot 3^n}$$

converge or diverge? Justify your answer.

Solution. Apply the ratio test:

$$\lim_{n\to\infty} \left[\frac{(n+1)^{n+1}}{(n+1)! \cdot 3^{n+1}} \div \frac{n^n}{n! \cdot 3^n} \right] = \lim_{n\to\infty} \frac{(n+1)^{n+1} n! \cdot 3^n}{n^n (n+1)! \cdot 3^{n+1}} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^n (n+1) \cdot 3}$$

$$= \frac{1}{3} \lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \frac{1}{3} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \frac{e}{3} < 1.$$

Hence, our series converges.

Problem # 4. Does the series

$$\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n} \cdot \ln n}$$

converge absolutely, converge conditionally, or diverge? Justify your answer.

Solution. Obviously, the sequence $\left\{ \frac{1}{\sqrt{n} \cdot \ln n} \right\}$ decreases and has limit 0. Hence, the series converges by the alternating series test. However, the series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n} \cdot \ln n}$ diverges by the limit comparison test with the harmonic series:

$$\lim_{n\to\infty} \left[\frac{1}{\sqrt{n} \cdot \ln n} \div \frac{1}{n} \right] = \lim_{n\to\infty} \frac{\sqrt{n}}{\ln n} = +\infty.$$

Hence, our series converges conditionally.

Problem # 5. For which values of x does the series

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n \cdot 2^n}$$

converge absolutely, converge conditionally, or diverge? Justify your answer.

Solution. Since $\lim_{n\to\infty} \left[\frac{1}{(n+1) \cdot 2^{n+1}} \div \frac{1}{n \cdot 2^n} \right] = \lim_{n\to\infty} \frac{n}{2(n+1)} = \frac{1}{2}$, the radius of convergence is 2. The interval of convergence is centered at $x = 1$. For $x = 3$, the series is $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges. For $x = 1$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges conditionally. Thus, our series converges absolutely for $-1 < x < 3$, converges contitionally for $x = -1$, and diverges for all other values of x.

Problem # 6. Find the sum of the series

$$\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \frac{\pi^9}{9!} - \dots$$

Solution. Since $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ for all x, our series is $\sin \pi = 0$.