LINEAR ALGEBRA (Prof. De Loera)
Graduate Proficiency Exam **Fall 2005**

These problems are intended to help you review material covered in the prelim. Some problems are harder and longer than what you will find in the exam. Try to write the solutions as you would in the exam: Write all details when solving every problem. Be organized and use the notation appropiately. Initially try to solve the problems without any assistance.

1. Given the matrix $A = \begin{bmatrix} 2 & -3 & -7 & 5 & 2 \\ 1 & -2 & -4 & 3 & 1 \\ 2 & 0 & -4 & 2 & 1 \\ 1 & -5 & -7 & 6 & 2 \end{bmatrix}$ find a basis for (a) $range(A)$, (b) $kernel(A)$.

   Determine whether the vector $[2, 2, -3, 7]$ is in the range of A or not.

2. Let $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$. Find the eigenvalues of A and if possible find a matrix S such that $S^{-1}AS$ is a diagonal matrix.

3. Can the matrix
   $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ be unitarily diagonalized? If yes, do it!

4. Given the matrix $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 6 & -11 & 6 \end{bmatrix}$ Find its eigenvalues, its characteristic polynomial and if possible diagonalize it (find explicit S such that $S^{-1}AS$ is diagonal).

5. The linear transformation given by derivation d/dt acts on the vector space of all polynomials with degree at most 3. Represent this linear transformation on the basis given by $\{1, t, t^2, t^3\}$. What is the Jordan canonical form of this matrix?

6. (a) What are the possible Jordan forms of an $n \times n$ matrix A such that $A^3 = I$?

   (b) What are the possible Jordan forms of a matrix $A \in M_6$ with characteristic polynomial $p_A(t) = (t+3)^4(t-2)^2$?

7. Apply the Gram-Schmidt procedure to the vectors $x_1 = (1, 1, 0)$, $x_2 = (2, 0, 1)$ and $x_3 = (2, 2, 1)$.

8. Prove or disprove: Let $A = \{x_1, x_2, \ldots, x_n\}$ be linearly independent vectors, then $A \cup \{y\}$ is a set of linearly independent vectors if and only if $y \notin span(A)$.

9. A linear transformation P is a projection if $P^2 = P$. Prove

   a) P is a projection if and only if $I - P$ is a projection.

b) $y \in range(P)$ if and only if $Py = y$.

c) $range(P) = kernel(I - P)$.

d) the vector space V is the direction sum of $range(P)$ and $kernel(P)$.

10. (a) If Λ denotes the diagonal $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Show that $\Lambda B = B\Lambda$ for some B, $n \times n$ matrix, if and only if B is also diagonal.

    (b) Suppose $A \in M_n$ has n distinct eigenvalues. If $AB = BA$ for some $B \in M_n$, show that A, B are simultaneously diagonalizable.

11. Let $A \in M_n$. If $rank(A) = 1$, then there exists a scalar $\beta \in \mathbb{C}$ such that $A^2 = \beta A$.

12. Given a 4×4 matrix of rank 3, explain which are the possible A-invariate subspaces of $\mathbb{C}^4$.

13. Are there real 3×3 matrix whose minimal polynomial is $x^2 + 1$? How about if the matrix is complex ? What happens if the matrix is 2×2?

14. True or false (justify):

    a) Similar matrices have the same minimal and characteristic polynomial

    b) If two matrices have the same minimal and characteristic polynomial they must be similar.

    c) inverse of an upper triangular matrix is upper triangular.

    d) Let W_1, W_2 be subspaces of V. Then $W_1 \cap W_2$ is a subspace too.

    e) Let W_1, W_2 be subspaces of V. Then $W_1 \cup W_2$ is a subspace too.

    f) Let A be a $k \times n$ matrix. $dim(range(A)) + dim(Kernel(A)) = n$

    g) Let $A \in M_{k,n}$ and $B \in M_{n,k}$. If $tr(X)$ denotes the trace of X, then $tr(AB) = tr(BA)$.

15. Let $A, B \in M_n$. Prove or disprove that

    a) $range(A + B) \subset range(A) + range(B)$.

    b) $range(AB) \subset range(A)$.

    c) $rank(AB) \leq rank(B)$.

    d) If A is non-singular then $rank(AB) = rank(B) = rank(BA)$.

16. Suppose $A, B \in M_n$ commute and have eigenvalues $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ respectively. Explain, with a proof, what are the eigenvalues of AB

17. a) State definitions of normal matrix, nilpotent matrix, and Hermitian matrix.

    b) Assume that A is normal and that $A, B \in M_n$ commute. Prove that A^* and B commute too.

    c)When is a normal matrix unitary? When is it Hermitian? When is it nilpotent?

    d) State three conditions equivalent to normality.

18. Let W, Y be subspaces of V such that $V = W + Y$. If $W \cap Y = \{0\}$, every vector $v \in V$ has a unique representation in the form $v = x + y$ with $x \in W,\ y \in Y$.

19. Let $A \in M_n$ and x, y represent non-zero vectors such that $Ax = 3x$ and $Ay = 5y$. Prove that x, y are linearly independent. How general is this fact?

20. Prove or disprove:

    a) $rank(A^2) \leq rank(A)$.

    b) $A, B \in M_n$ with AB diagonalizable implies BA diagonalizable.

    c) If P is a projection its eigenvalues are only $0, 1$.

    d) Let P, Q be projections which are simultaneously diagonalizable, then PQ is a projection too.

21. Sally is playing "matrix of fortune" and knows that $4, -7, 2$ are eigenvalues of the following matrix $\begin{bmatrix} 2 & * & * & 6 \\ * & 3 & * & * \\ * & * & -1 & * \\ * & * & * & 5 \end{bmatrix}$ To win the game she needs to find $det(A)$. Please help her!

22. Prove or disprove (with justification):

    a) If A, B are two $n \times n$ matrices and $A = A^t$ and $B = B^t$ then $AB = (AB)^t$.

    b) If A is nilpotent matrix then 0 is the only eigenvalue of A.

    c) Similar matrices have the same eigenvalues.

    d) If λ is an eigenvalue of both A, B then λ is an eigenvalue of $A + B$.

    e) Matrices with the same determinant have the same rank.

    f) If λ is an eigenvalue of an orthogonal transformation then $\lambda = \pm 1$

    g) Matrices with the same characteristic polynomial are similar.

    h) The set of all $n \times n$ real matrices with determinant zero is a subspace of M_n.

    i) If a square matrix A is diagonalizable so is A^2.

    j) Similar matrices have the same eigenvectors.

    k) If V, W are two distinct 2-dimensional subspaces of R^3, then $V + W = R^3$.

    l) Let $\{x, y, z\}$ be a linearly independent set of vectors in a vector space over a finite field. The set $\{x + y, x + z, y + z\}$ is always linearly independent.

23. Let V and W be the following subspaces of $\mathbb{R}^4$.

    $V = span\{(1, 3, -4, 7), (-1, 2, -1, 3)\}$, $W = span\{(2, 4, 3, 1), (2, 2, -1, 3)\}$.

    Determine bases for the subspaces $V + W$, $W \cap V$, and $V^\perp$.

24. Given the matrix $A = \begin{bmatrix} 21 & 6 & -12 \\ 6 & 12 & -6 \\ -12 & -6 & 21 \end{bmatrix}$

    Find the eigenvalues and an orthonormal basis for each of the eigenspaces. Is this matrix diagonalizable over R? Can you find a matrix B, such that $B^2 = A$.?

25. Determine whether the following matrices are diagonalizable over the indicated field. If possible find the diagonalization.

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix}$$ over the real numbers

$$\begin{bmatrix} 3 & 0 & 1 \\ -1 & 2 & -1 \\ 1 & 2 & 5 \end{bmatrix}$$ over the real numbers

$$\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$$ over the complex numbers.

26. Let A be a normal and idempotent matrix. Then $\sum_{i,j} |a_{ij}|^2 = rank(A)$.

27. Let $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$. Find the Jordan canonical form of A.

28. Let $A = \begin{bmatrix} 3 & -3 & 6 \\ -3 & 2 & -9 \\ 6 & -9 & 2 \end{bmatrix}$.

    Is this matrix diagonalizable? If yes, please find the diagonalization. Show that A determines an indefinite quadratic form. Determine vectors x, y such that $x^t A x > 0$ and $y^t A y < 0$. How many eigenvalues are negative, positive, or zero?

29. For an $m \times n$ matrix A:

    Prove that the following conditions are equivalent a) $rank(A) = n$, b) $A^t A$ is invertible, c) $BA = I$ for some $n \times m$ matrix B.

    Prove that the eigenvalues of $A^t A$ are non-negative and the same as those of AA^t.

30. Are the eigenspaces of a matrix always orthogonal to each other?