# Convex Integer Maximization via Graver Bases

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### Abstract

We present a new algebraic algorithmic scheme to solve *convex integer maximization* problems of the following form, where $c$ is a convex function on $\mathbb{R}^d$ and $w_1 x, \ldots, w_d x$ are linear forms on $\mathbb{R}^n$,

$$\max \{ c(w_1 x, \ldots, w_d x) : \ Ax = b, \ x \in \mathbb{N}^n \} \ .$$

This method works for arbitrary input data $A, b, d, w_1, \ldots, w_d, c$. Moreover, for fixed $d$ and several important classes of programs in *variable dimension*, we prove that our algorithm runs in *polynomial time*. As a consequence, we obtain polynomial time algorithms for various types of multi-way transportation problems, packing problems, and partitioning problems in variable dimension.

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## 1   Introduction

In the past fifteen years algebraic geometry and commutative algebra tools have shown their exciting potential to study problems in integer optimization (see [5, 30] and references therein). But, so far, algebraic methods have always been considered "guilty" of bad computational complexity, namely, the notorious bad complexity for computing general Gröbner bases when the number of variables grow (see [22] and references therein). This paper demonstrates that, by carefully analyzing the structure of toric ideals in particular problems, algebraic tools can compete (and win!) against more mainstream tools in optimization.

The main algebraic ingredient we will need is the notion of *Graver bases*, a special kind of universal Gröbner bases for the toric ideals associated with integer matrices. We recommend the introduction

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presented in Chapter 4 of [28] for a basic introduction to Gröbner and Graver bases of toric ideals. We consider a new algorithmic scheme for solving the following far-reaching generalization of standard linear integer programming:

**Convex Integer Maximization**. Given positive integers $d, m, n$, integer vectors $w_1, \ldots, w_d \in \mathbb{Z}^n$ and $b \in \mathbb{Z}^m$, integer matrix $A \in \mathbb{Z}^{m \times n}$, and convex function $c : \mathbb{R}^d \longrightarrow \mathbb{R}$, find a nonnegative integer vector $x \in \mathbb{N}^n$ maximizing the objective function $c(w_1 x, \ldots, w_d x)$ subject to the equation system $Ax = b$,

$$\max \left\{ c(w_1 x, \ldots, w_d x) : \ Ax = b, \ x \in \mathbb{N}^n \right\} .$$

This problem can be interpreted as multi-objective integer programming: given $d$ different linear objective functions $w_1, \ldots, w_d$, the goal is to maximize their "convex balancing" given by $c(w_1 x, \ldots, w_d x)$. The convex integer maximization problem is very expressive and in fact, contains a whole hierarchy of problems of increasing generality and complexity, parameterized by the number $d$ of linear objectives used: at the bottom lies the standard linear integer programming problem, recovered as the special case of $d = 1$ and $c$ the identity on $\mathbb{R}$; and at the top lies the problem of maximizing an arbitrary convex functional over the set of integer points in a rational polyhedron in $\mathbb{R}^n$, arising with $d = n$ and $w_i = \mathbf{1}_i$ the $i$-th standard unit vector in $\mathbb{R}^n$ for all $i$.

In general, convex integer maximization is intractable even for small fixed $d$, since already for $d = 1$ it includes linear integer programming which is NP-hard. For variable $d$, even very simple special cases are NP-hard, such as the following instance (*positive semi-definite quadratic binary programming*),

$$\max \left\{ (w_1 x)^2 + \cdots + (w_n x)^2 : \ x_i + y_i = 1 \ \ (i = 1, \ldots, n), \ \ x, y \in \mathbb{N}^n \right\} .$$

Clearly, the complexity of the problem depends also on the presentation of the convex function: we will assume that $c$ is presented by a *comparison oracle* that, queried on $x, y \in \mathbb{R}^d$, asserts whether or not $c(x) \leq c(y)$. This is a very broad presentation that reveals little information on the function, making the problem harder to solve. In particular, if the polyhedron $\{ x \in \mathbb{R}^n_+ : \ Ax = b \}$ is unbounded, then the problem is inaccessible even in one variable with no equation constraints: consider the following family of univariate convex integer programs with convex functions parameterized by $-\infty < u \leq \infty$,

$$\max \left\{ c_u(x) : \ x \in \mathbb{N} \right\} , \quad c_u(x) := \begin{cases} -x, & \text{if } x < u; \\ x - 2u, & \text{if } x \geq u. \end{cases} ;$$

now consider any algorithm attempting to solve the problem and let $u$ be the maximum value of $x$ in all queries to the oracle of $c$; then the algorithm can not distinguish between the problem with $c_u$, whose objective function is unbounded, and the problem with $c_\infty$, whose optimal objective value is 0. (We remark that, for explicitly given (rather than oracle presented) simple convex functions, it might be possible to handle unbounded feasible regions as well; this should be the subject of future study.)

In spite of these difficulties, we show in this article that the algebraic techniques of Graver bases allow us to solve the convex integer maximization problem in polynomial time for a large and useful

class of integer programs in variable dimension. Moreover, this class is *universal* for integer programming in a well defined sense, enabling to extend this to an algorithmic scheme for solving convex integer maximization over arbitrary integer programs.

Our first key lemma, extending results of [23] for combinatorial optimization, shows that when a suitable geometric condition holds, it is possible to efficiently reduce the convex integer maximization problem to the solution of polynomially many linear integer programming counterparts. As we will see, this condition holds naturally for a broad class of problems in variable dimension. To state this result, we need the following terminology. A *direction* of an edge (1-face) $e$ of a polyhedron $P$ is an nonzero scalar multiple of $u - v$ with $u, v$ any two distinct points in $e$. A set of vectors *covers all edge-directions of $P$* if it contains a direction of each edge of $P$. A *linear integer programming oracle* for matrix $A \in \mathbb{Z}^{m \times n}$ and vector $b \in \mathbb{Z}^m$ is one that, queried on $w \in \mathbb{Z}^n$, solves the linear integer program $\max\{wx : Ax = b, \ x \in \mathbb{N}^n\}$, that is, either returns an optimal solution $x \in \mathbb{N}^n$, or asserts that the program is infeasible, or asserts that the objective function $w$ is unbounded.

**Lemma 1.1** *For any fixed $d$ there is a strongly polynomial oracle-time algorithm that, given any vectors $w_1, \ldots, w_d \in \mathbb{Z}^n$, matrix $A \in \mathbb{Z}^{m \times n}$ and vector $b \in \mathbb{Z}^m$ endowed with a linear integer programming oracle, finite set $E \subset \mathbb{Z}^n$ covering all edge-directions of the polyhedron $\mathrm{conv}\{x \in \mathbb{N}^n : Ax = b\}$, and convex functional $c : \mathbb{R}^d \longrightarrow \mathbb{R}$ presented by a comparison oracle, solves the convex integer program*

$$\max\left\{c(w_1 x, \ldots, w_d x) : \ Ax = b, \ x \in \mathbb{N}^n\right\}.$$

Here, *solving* the program means that the algorithm either returns an optimal solution $x \in \mathbb{N}^n$, or asserts the problem is infeasible, or asserts the polyhedron $\{x \in \mathbb{R}^n_+ : Ax = b\}$ is unbounded in which case the problem is hopeless (see discussion above); and *strongly polynomial oracle-time* means that the number of arithmetic operations and calls to the oracles are polynomially bounded in $m$ and $n$, and the size of the numbers occurring throughout the algorithm is polynomially bounded in the size of the input (which is the number of bits in the binary representation of the entries of $w_1, \ldots, w_d, A, b, E$).

Our main theorem, building on Lemma 1.1, shows that a broad (in fact, *universal*) class of convex integer maximization problems can be solved in polynomial time. Given an $(r + s) \times t$ matrix $A$, let $A_1$ be its $r \times t$ sub-matrix consisting of the first $r$ rows and let $A_2$ be its $s \times t$ sub-matrix consisting of the last $s$ rows. Define the *n-fold matrix* of $A$ to be the following $(r + ns) \times nt$ matrix,

$$A^{(n)} \quad := \quad (\mathbf{1}_n \otimes A_1) \oplus (I_n \otimes A_2) \quad = \quad \begin{pmatrix} A_1 & A_1 & A_1 & \cdots & A_1 \\ A_2 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_2 \end{pmatrix}.$$

Note that $A^{(n)}$ depends on $r$ and $s$: these will be indicated by referring to $A$ as an "$(r + s) \times t$ matrix".

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We establish the following theorem, which asserts that convex integer maximization over $n$-fold systems of a fixed matrix $A$, in variable dimension $nt$, are solvable in polynomial time. This extends results for *linear* integer programming from [12].

**Theorem 1.2** *For any fixed positive integer $d$ and fixed $(r+s) \times t$ integer matrix $A$ there is a polynomial oracle-time algorithm that, given $n$, vectors $w_1, \ldots, w_d \in \mathbb{Z}^{nt}$ and $b \in \mathbb{Z}^{r+ns}$, and convex function $c : \mathbb{R}^d \longrightarrow \mathbb{R}$ presented by a comparison oracle, solves the convex $n$-fold integer maximization problem*

$$\max \{ c(w_1 x, \ldots, w_d x) : \ A^{(n)} x = b, \ x \in \mathbb{N}^{nt} \} \ .$$

The equations defined by an n-fold matrix have the following, perhaps more illuminating, interpretation: splitting the variable vector and the right-hand side vector into components of suitable sizes, $x = (x^1, \ldots, x^n)$ and $b = (b^0, b^1, \ldots, b^n)$, where $b^0 \in \mathbb{Z}^r$ and $x^k \in \mathbb{N}^t$ and $b^k \in \mathbb{Z}^s$ for $k = 1, \ldots, n$, the equations become $A_1(\sum_{k=1}^n x^k) = b^0$ and $A_2 x^k = b^k$ for $k = 1, \ldots, n$. Thus, each component $x^k$ satisfies a system of constraints defined by $A_2$ with its own right-hand side $b^k$, and the sum $\sum_{k=1}^n x^k$ obeys constraints determined by $A_1$ and $b^0$ restricting the "common resources shared by all components".

Theorem 1.2 has various applications, including to multiway transportation problems, packing problems, vector partitioning and clustering, which will be discussed in Section 3. For example, we have the following corollary providing the first polynomial time solution of convex 3-way transportation.

**Corollary 1.3 (convex 3-way transportation)** *For any fixed $d, p, q$ there is a polynomial oracle-time algorithm that, given $n$, arrays $w_1, \ldots, w_d \in \mathbb{Z}^{p \times q \times n}$, $u \in \mathbb{Z}^{p \times q}$, $v \in \mathbb{Z}^{p \times n}$, $z \in \mathbb{Z}^{q \times n}$, and convex $c : \mathbb{R}^d \longrightarrow \mathbb{R}$ presented by comparison oracle, solves the convex integer 3-way transportation problem*

$$\max \{ c(w_1 x, \ldots, w_d x) \ : \ x \in \mathbb{N}^{p \times q \times n}, \ \sum_i x_{i,j,k} = z_{j,k}, \ \sum_j x_{i,j,k} = v_{i,k}, \ \sum_k x_{i,j,k} = u_{i,j} \} \ .$$

Note that in contrast, if the dimensions of two sides of the tables are variable, say, $q$ and $n$, then even the standard *linear* integer 3-way transportation problem over such tables is NP-hard, see [9, 10, 11].

We proceed to discuss the universality of $n$-fold integer programming and describe our algorithmic scheme for solving convex integer maximization over an arbitrary system. Define a variant of the $n$-fold operator as follows: for an $s \times t$ matrix $A$, define its $n$-*product* $A^{[n]}$ to be the $n$-fold product of the $(t+s) \times t$ matrix obtained by appending $A$ to the $t \times t$ identity matrix $I_t$, that is:

$$A^{[n]} \ := \ \left( \begin{array}{c} I_t \\ A \end{array} \right)^{(n)} \ .$$

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Consider $m$-products $(1,1,1)^{[m]}$ of the $1 \times 3$ matrix $(1,1,1)$. Note that $(1,1,1)^{[m]}$ is precisely the $(3+m) \times 3m$ vertex-edge incidence matrix of the complete bipartite graph $K_{3,m}$. For instance,

$$(1,1,1)^{[3]} \;=\; \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} .$$

The following result which incorporates the recent universality theory of [9, 10, 11] asserts that *every* convex integer maximization problem can be lifted in polynomial time to some convex integer maximization problem defined by some $n$-product of some $m$-product of $(1,1,1)$. Theorem 1.2 can then be harnessed to solve the lifted program, providing a general solution scheme for convex maximization.

**Corollary 1.4 (scheme for arbitrary convex integer maximization)** *There is a polynomial time algorithm that, given integer $p \times q$ matrix $B$ and $b \in \mathbb{Z}^p$ with $\{x \in \mathbb{Z}^q : Bx = b, \ x \geq 0\}$ bounded, computes $m$, $n$, and integer $(3m + n(3+m))$ vector $a$ such that, for any given vectors $w_1, \ldots w_d \in \mathbb{Z}^q$ and any convex function $c$ on $\mathbb{R}^d$, the corresponding convex integer maximization problem lifts to a convex integer maximization problem defined by the $n$-product of the $m$-product of $(1,1,1)$, that is,*

$$\max\{c(w_1 x, \ldots, w_d x) : x \in \mathbb{Z}^q, \ Bx = b, \ x \geq 0\}$$
$$= \; \max\left\{c(\hat{w}_1 \hat{x}, \ldots, \hat{w}_d \hat{x}) \;:\; \hat{x} \in \mathbb{Z}^{3mn}, \; \left((1,1,1)^{[m]}\right)^{[n]} \hat{x} \;=\; a, \; \hat{x} \geq 0\right\} .$$

*The algorithm also computes an embedding of $\mathbb{Z}^q$ into $\mathbb{Z}^{3mn}$ so that the vectors $\hat{w}_1, \ldots, \hat{w}_d \in \mathbb{Z}^{3mn}$ are obtained from the corresponding vectors $w_1, \ldots w_d \in \mathbb{Z}^q$ by simply adding sufficiently many 0 entries.*

*Proof.* Reformulating the universality theorem for multiway tables from [10] in terms of products, it asserts that the set of integer points $\{x \in \mathbb{Z}^q : Bx = b, \ x \geq 0\}$ in any rational polytope stands in polynomial-time computable coordinate-embedding linear bijection with the set of integer points in the polytope $\left\{\hat{x} \in \mathbb{Z}^{3mn} : \left((1,1,1)^{[m]}\right)^{[n]} \hat{x} \;=\; a, \ \hat{x} \geq 0\right\}$ for some $m$, $n$ and $a$. Lifting each $w_i \in \mathbb{Z}^q$ to $\hat{w}_i \in \mathbb{Z}^{3mn}$ by adding suitable 0 entries, implies that for every integer point $x$ in the original program and its corresponding integer point $\hat{x}$ in the lifted program, we have the same objective function value $c(w_1 x, \ldots, w_d x) = c(\hat{w}_1 \hat{x}, \ldots, \hat{w}_d \hat{x})$. Thus, the optimal objective function values in the original and lifted programs are the same, and, moreover, an optimal solution to the original program can be read off as any point $x$ corresponding to any optimal solution $\hat{x}$ to the lifted program. $\quad\square$

Note that, if P$\neq$NP, there can be no polynomial time algorithm for general linear integer programming, let alone convex integer maximization. So how does this reconcile with the scheme suggested by Corollary 1.4 above ? The point is that, for every fixed $m$, Theorem 1.2 provides a polynomial time

algorithm for convex maximization over all integer programs that lift to programs with defining matrix that is the $n$-product $\left((1,1,1)^{[m]}\right)^{[n]}$ of $(1,1,1)^{[m]}$. But for arbitrary integer programs, $m$ is variable as well and so the whole procedure is not polynomial. But in practice, this might be efficient or enable a quick approximation, and should be the subject of future study. We also note that, for fixed $m$, the computational complexity of solving convex maximization over programs defined by $\left((1,1,1)^{[m]}\right)^{[n]}$ is dominated by $n^{dg(m)}$, where $g(m)$ is the so-called *Graver complexity* of the complete bipartite graph $K_{3,m}$ and of its incidence matrix $(1,1,1)^{[m]}$. The precise rate of growth of $g(m)$ as a function of $m$ is unknown and intriguing; see [4] for the best bounds and for more details and precise definitions.

The rest of the article proceeds as follows. In Section 2 we give the proofs of all statements. We begin by discussing edge-directions of polyhedra and provide the algorithm establishing Lemma 1.1. We proceed to discuss Graver bases and, incorporating Lemma 1.1 and recent results from [12], which are based on results of Hoşten and Sullivant [21] and Santos and Sturmfels [27] on the asymptotic stabilization of Graver bases, we are able to establish Theorem 1.2. In Section 3 we discuss applications to multiway transportation, packing, vector partitioning and clustering, as follows: in 3.1 we obtain Corollary 1.3 and an extension to $k$-way transportation problems of any dimension $k$ (Corollary 3.1); in 3.2 we describe applications to bin packing problems (Corollary 3.2); finally, in 3.3 we apply our Theorem 1.2 to vector partitioning in general and clustering in particular (Corollary 3.4).

## 2  Proofs

In this section we prove Lemma 1.1, which is of interest in its own right, and combine it with several other results to establish our main Theorem 1.2. Before proceeding with the details, we provide the main outline and point out the difficulties that we have to overcome. Given data for a convex integer maximization problem $\max\{c(w_1 x, \ldots, w_d x) : \ Ax = b, \ x \in \mathbb{N}^n\}$, consider the polyhedron $P := \mathrm{conv}\{x \in \mathbb{N}^n : Ax = b\} \subseteq \mathbb{R}^n$ and its projection $Q := \{(w_1 x, \ldots, w_d x) : \ x \in P\} \subseteq \mathbb{R}^d$. Note that $P$ is the so-called *integer hull* of $\{x \in \mathbb{R}^n : Ax = b, \ x \geq 0\}$ and has typically exponentially many vertices and is not accessible computationally. Note also that, since $c$ is convex, there is an optimal solution $x$ whose projection $(w_1 x, \ldots, w_d x)$ is a vertex of $Q$. So an important ingredient in the solution is to construct the vertices of $Q$. Unfortunately, $Q$ may also have exponentially many vertices even though it lives in a space $\mathbb{R}^d$ of fixed dimension. However, we will be able to show that, when the number of *edge-directions* of $P$ is polynomial, the number of vertices of $Q$ is polynomial. Nonetheless, even in this case, it is not possible to construct these vertices directly, since the number of vertices of $P$ may still be exponential. To overcome this difficulty, we need to make use of a suitable *zonotope*. This is the key idea underlying the algorithm of Lemma 1.1. Next, we restrict attention to $n$-fold systems. For such systems, using recent results of [21, 27] on the stabilization of their *Graver bases*, we are able to show that the set of edge-directions of the integer hull $P$ can be computed in polynomial time. Combining this with Lemma 1.1 and several other results from [12] we obtain Theorem 1.2.

We now proceed with the precise details. As defined earlier, a *direction* of an edge (1-face) $e$ of a polyhedron $P$ is any nonzero scalar multiple of $u - v$ where $u, v$ are any two distinct points in $e$. We say that a set of vectors $E$ *covers all edge-directions of* $P$ if it contains a direction of each edge of $P$. A polyhedron $Z$ is a *refinement* of a polyhedron $P$ if the closure of each normal cone of $P$ is the union of closures of normal cones of $Z$. The *zonotope* generated by a finite set $E \subset \mathbb{R}^n$ is the polytope $Z := \mathrm{zone}(E) := \mathrm{conv}\{\sum_{e \in E} \lambda_e e : \lambda_e = \pm 1\}$. More details and proofs of the next two propositions can be found in [16, 23, 24] and the references therein.

**Proposition 2.1** *Let $E \subset \mathbb{R}^n$ be a finite set covering all edge-directions of a polyhedron $P \subseteq \mathbb{R}^n$. Then the* zonotope $Z := \mathrm{zone}(E) = \mathrm{conv}\{\sum_{e \in E} \lambda_e e : \lambda_e = \pm 1\}$ *generated by $E$ is a refinement of $P$.*

**Proposition 2.2** *For any fixed $d$, there is a polynomial time algorithm that, given any $E \subset \mathbb{Z}^d$, outputs every vertex $v$ of $Z := \mathrm{zone}(E)$ along with $g_v \in \mathbb{Z}^d$ with $g_v x$ uniquely maximized over $Z$ at $v$.*

We can now prove Lemma 1.1, showing that a set of edge-directions of the polyhedron underlying a convex integer program allows to solve it by solving polynomially many linear integer counterparts.

**Lemma 1.1** *For any fixed $d$ there is a strongly polynomial oracle-time algorithm that, given any vectors $w_1, \ldots, w_d \in \mathbb{Z}^n$, matrix $A \in \mathbb{Z}^{m \times n}$ and vector $b \in \mathbb{Z}^m$ endowed with a linear integer programming oracle, finite set $E \subset \mathbb{Z}^n$ covering all edge-directions of the polyhedron $\mathrm{conv}\{x \in \mathbb{N}^n : Ax = b\}$, and convex functional $c : \mathbb{R}^d \longrightarrow \mathbb{R}$ presented by a comparison oracle, solves the convex integer program*

$$\max\{c(w_1 x, \ldots, w_d x) : Ax = b, \ x \in \mathbb{N}^n\}.$$

*Proof.* We provide the algorithm claimed by the theorem. First, query the linear integer programming oracle of $A, b$ on the trivial linear function $w = 0$; if the oracle asserts that the linear problem is infeasible, then terminate the algorithm asserting that the convex problem is infeasible. So assume the problem is feasible. Let $P := \mathrm{conv}\{x \in \mathbb{N}^n : Ax = b\} \subseteq \mathbb{R}^n$ and $Q := \{(w_1 x, \ldots, w_d x) : x \in P\} \subseteq \mathbb{R}^d$. Then $Q$ is a projection of $P$, and the corresponding projection $D := \{(w_1 e, \ldots, w_d e) : e \in E\}$ of the set $E$ is a set covering all edge-directions of $Q$. Let $Z := \mathrm{zone}(D) \subset \mathbb{R}^d$ be the zonotope generated by $D$. Since $d$ is fixed, by Proposition 2.2 we can produce in polynomial time all vertices of $Z$, every vertex $v$ along with $g_v \in \mathbb{Z}^d$ such that the linear function defined by $g_v$ is uniquely maximized over $Z$ at $v$. For each of the polynomially many $g_v$, repeat the following procedure. Define a vector $h_v \in \mathbb{Z}^n$ by $h_{v,j} := \sum_{i=1}^{d} w_{i,j} g_{v,i}$ for $j = 1, \ldots, n$. Now query the linear integer programming oracle of $A, b$ on the linear function $w := h_v \in \mathbb{Z}^n$. If the oracle replies that the objective is unbounded, then terminate the algorithm asserting that $P$ is an unbounded polyhedron. Otherwise, let $x_v \in P \cap \mathbb{N}^n$ be the optimal solution obtained from the oracle, and let $z_v := (w_1 x_v, \ldots, w_d x_v) \in Q$ be its projection. Since for every $x \in P$ and its projection $z := (w_1 x, \ldots, w_d x) \in Q$ we have $g_v z = h_v x$, we conclude that $z_v$ is a

maximizer of $g_v$ over $Q$. Now we claim that each vertex $u$ of $Q$ equals some $z_v$. Indeed, since $Z$ is a refinement of $Q$ by Proposition 2.1, it follows that there is some vertex $v$ of $Z$ such that $g_v$ is uniquely maximized over $Q$ at $u$, and therefore $u = z_v$. Suppose that the linear integer programming oracle replied with an optimal solution to each query. Since $Z$ refines $Q$, this implies that $Q$ is bounded hence a polytope. Since $c(w_1 x, \ldots, w_d x)$ is convex on $\mathbb{R}^n$ and $c$ is convex on $\mathbb{R}^d$, we have that

$$\max\{c(w_1 x, \ldots, w_d x) : Ax = b, \ x \in \mathbb{N}^n\} \ = \ \max\{c(w_1 x, \ldots, w_d x) : x \in P\}$$

$$= \ \max\{c(z) : z \in Q\} \ = \ \max\{c(u) : u \text{ vertex of } Q\} \ = \ \max\{c(z_v) : v \text{ vertex of } Z\} \ .$$

Using the comparison oracle for $c$, identify that $z_v$ achieving maximum value $c(z_v)$ over all vertices $v$ of $Z$, and output $x_v$ which is the optimal solution to the convex integer programming problem.    $\square$

Recall that solving the convex integer program means that the algorithm either returns an optimal solution $x \in \mathbb{N}^n$, or asserts that the problem is infeasible, or asserts that the polyhedron $\{x \in \mathbb{R}^n \ : \ Ax = b\}$ is unbounded in which case the problem is generally hopeless (see discussion in the introduction). It may happen, though, that the projection $Q$ of $P$ is bounded even though $P$ is not: in this case, there is an optimal solution to the convex integer programming problem, and our algorithm *will* find it.

Lemma 1.1 bares at once useful consequences for systems whose defining matrix $A$ is totally unimodular, such as network flow problems and ordinary (2-way) transportation problems. For such totally unimodular systems, the relevant polyhedron $P$ is integer, that is, we have the equality

$$P \ := \ \mathrm{conv}\{x \in \mathbb{N}^n : Ax = b\} \ = \ \{x \in \mathbb{R}^n_+ : Ax = b\} \ := \ L \ .$$

This implies the following two useful properties: first, for any integer vector $b$, a linear integer programming oracle for $A, b$ is polynomial time realizable by linear programming over $L$; and second, a set $E$ covering all edge-directions of $P$ is provided by the set of *circuits* of $A$, that is, minimal-support linear dependencies on the columns of $A$, whose cardinality is bounded above by $\binom{n}{m}$. If $m$ grows slowly, say $m = O(\log n)$, then this bound is sub-exponential and the algorithm underlying Lemma 1.1 might provide a good strategy for addressing the convex integer maximization problem.

Next, we proceed to prove Theorem 1.2. We need to recall some definitions. The *Graver basis* of an integer matrix $A$, introduced in [15], is a canonical finite set $\mathcal{G}(A)$ that can be defined as follows. Let $\mathcal{L}(A) := \{x \in \mathbb{Z}^n : \ Ax = 0\}$ be the lattice of integer linear dependencies on $A$. Define a partial order $\sqsubseteq$ on $\mathbb{Z}^n$ which extends the coordinate-wise order $\leq$ on $\mathbb{N}^n$ as follows: for two vectors $u, v \in \mathbb{Z}^n$ put $u \sqsubseteq v$ and say that $u$ is *conformal* to $v$ if $|u_i| \leq |v_i|$ and $u_i v_i \geq 0$ for $i = 1, \ldots, n$, that is, $u$ and $v$ lie in the same orthant of $\mathbb{R}^n$ and each component of $u$ is bounded by the corresponding component of $v$ in absolute value. The Graver basis of $A$ is then the set $\mathcal{G}(A)$ of all $\sqsubseteq$-minimal vectors in $\mathcal{L}(A) \setminus \{0\}$. For instance, if $A = (1, 2, 1)$ then $\mathcal{G}(A) = \pm\{(2, -1, 0), (0, -1, 2), (1, 0, -1), (1, -1, 1)\}$. For more details on Graver bases and the currently fastest procedure for computing them see [28, 17, 18].

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It is known that the universal Gröbner bases of $A$, namely the union of all reduced Gröbner bases of the toric ideal of the matrix $A$, contains all edge directions in the integer hulls within the polytopes $P_b = \{x : Ax = b, x \geq 0\}$ (see Section 5 in [29]. Since the Graver bases contains this universal one can deduce the following property (we include a direct proof here):

**Lemma 2.3** *For every integer matrix $A \in \mathbb{Z}^{m \times n}$ and every integer vector $b \in \mathbb{N}^m$, the Graver basis $\mathcal{G}(A)$ of $A$ covers all edge-directions of the polyhedron $\mathrm{conv}\{x \in \mathbb{N}^n : Ax = b\}$ defined by $A$ and $b$.*

*Proof.* Consider any edge $e$ of $P := \mathrm{conv}\{x \in \mathbb{N}^n : Ax = b\}$ and pick two distinct points $u, v \in e \cap \mathbb{N}^n$. Then $g := u - v$ is in $\mathcal{L}(A) \setminus \{0\}$ and hence $g$ is a *conformal sum* $g = \sum g^i$ with $g^i \sqsubseteq g$ and $g^i \in \mathcal{G}(A)$ for all $i$. To see this, recall that $\mathcal{G}(A)$ is the set of $\sqsubseteq$-minimal elements in $\mathcal{L}(A) \setminus \{0\}$ and note that $\sqsubseteq$ is a well-ordering; if $g \in \mathcal{G}(A)$, we are done; otherwise there is an $h \in \mathcal{G}(A)$ with $h \sqsubset g$ in which case, by induction on $\sqsubseteq$, there is a conformal sum $g - h = \sum g^i$ giving the conformal sum $g = h + \sum g^i$.

Now, we claim that $u - g^i \in P$ for all $i$. To see this, note first that $g^i \in \mathcal{G}(A) \subset \mathcal{L}(A)$ implies $Ag^i = 0$ and hence $A(u - g^i) = Au = b$; and second, note that $u - g^i \geq 0$: indeed, if $g^i_j \leq 0$ then $u_j - g^i_j \geq u_j \geq 0$; and if $g^i_j > 0$ then $g^i \sqsubseteq g$ implies $g^i_j \leq g_j$ and therefore $u_j - g^i_j \geq u_j - g_j = v_j \geq 0$.

Now let $w \in \mathbb{R}^n$ be a linear functional uniquely maximized over $P$ at the edge $e$. Then for all $i$, as just proved, $u - g^i \in P$ and hence $wg^i \geq 0$. But $\sum wg^i = wg = wu - wv = 0$, implying that in fact, for all $i$, we have $wg^i = 0$ and therefore $u - g^i \in e$. This implies that each $g^i$ is a direction of the edge $e$ (in fact, moreover, all $g^i$ are the same, so $g$ is a multiple of some Graver basis element). $\square$

We also need the following two recent results from [12] on n-fold systems. The first result builds on stabilization of Graver bases established by Hoşten and Sullivant [21] and Santos and Sturmfels [27].

**Proposition 2.4** *For any fixed $(r+s) \times t$ integer matrix $A$ there is a polynomial time algorithm that, given any $n$, computes the Graver basis $\mathcal{G}(A^{(n)})$ of the n-fold matrix $A^{(n)} = (\mathbf{1}_n \otimes A_1) \oplus (I_n \otimes A_2)$.*

The second result of [12] combines Proposition 2.4 and the use of the Graver basis for augmentation.

**Proposition 2.5** *For any fixed $(r+s) \times t$ integer matrix $A$ there is a polynomial time algorithm that, given $n$ and vectors $w \in \mathbb{Z}^{nt}$ and $b \in \mathbb{Z}^{r+ns}$, solves the linear n-fold integer programming problem*

$$\max \{wx : A^{(n)}x = b, \ x \in \mathbb{N}^{nt}\} .$$

Combining Lemma 1.1, Lemma 2.3, and Propositions 2.4 and 2.5, we can now prove Theorem 1.2.

**Theorem 1.2** *For any fixed positive integer $d$ and fixed $(r+s) \times t$ integer matrix $A$ there is a polynomial oracle-time algorithm that, given $n$, vectors $w_1, \ldots, w_d \in \mathbb{Z}^{nt}$ and $b \in \mathbb{Z}^{r+ns}$, and convex function $c : \mathbb{R}^d \longrightarrow \mathbb{R}$ presented by a comparison oracle, solves the convex n-fold integer maximization problem*

$$\max \{c(w_1 x, \ldots, w_d x) : A^{(n)}x = b, \ x \in \mathbb{N}^{nt}\} .$$

*Proof.* The algorithm underlying Proposition 2.5 provides a polynomial time realization of a linear integer programming oracle for $A^{(n)}$ and $b$. The algorithm underlying Proposition 2.4 allows to compute the Graver basis $\mathcal{G}(A^{(n)})$ in time which is polynomial in the input. By Lemma 2.3, this set $E := \mathcal{G}(A^{(n)})$ covers all edge-directions of the polyhedron $\mathrm{conv}\{x \in \mathbb{N}^{nt} : A^{(n)}x = b\}$ underlying the convex integer program. Thus, the hypothesis of Lemma 1.1 is satisfied and hence the algorithm underlying Lemma 1.1 can be used to solve the convex integer maximization problem in polynomial time. $\qquad\square$

## 3   Applications

We now discuss various applications of our results to multiway transportation problems, packing problems, vector partitioning and clustering, extending and unifying applications from [12, 19, 23, 24].

### 3.1   Multiway transportation problems

A $k$-way transportation polytope is the set of all $m_1 \times \cdots \times m_k$ nonnegative arrays $x = (x_{i_1,\ldots,i_k})$ such that the sums of the entries over some of their lower dimensional sub-arrays (margins) are specified. More precisely, for any tuple $(i_1,\ldots,i_k)$ with $i_j \in \{1,\ldots,m_j\} \cup \{+\}$, the corresponding *margin* $x_{i_1,\ldots,i_k}$ is the sum of entries of $x$ over all coordinates $j$ with $i_j = +$. The *support* of $(i_1,\ldots,i_k)$ and of $x_{i_1,\ldots,i_k}$ is the set $\mathrm{supp}(i_1,\ldots,i_k) := \{j : i_j \neq +\}$ of non-summed coordinates. For instance, if $x$ is a $4 \times 5 \times 3 \times 2$ array then it has 12 margins with support $F = \{1,3\}$ such as $x_{3,+,2,+} = \sum_{i_2=1}^{5} \sum_{i_4=1}^{2} x_{3,i_2,2,i_4}$. Given a family $\mathcal{F}$ of subsets of $\{1,\ldots,k\}$ and margin values $u_{i_1,\ldots,i_k}$ for all tuples with support in $\mathcal{F}$, the corresponding $k$-way transportation polytope is the set of nonnegative arrays with these margins,

$$T_{\mathcal{F}} \;=\; \left\{ x \in \mathbb{R}_+^{m_1 \times \cdots \times m_k} \;:\; x_{i_1,\ldots,i_k} = u_{i_1,\ldots,i_k}, \;\; \mathrm{supp}(i_1,\ldots,i_k) \in \mathcal{F} \right\} \;.$$

Transportation polytopes and their integer points (called contingency tables by statisticians), have been studied and used extensively in the operations research literature and in the context of secure statistical data disclosure by public agencies, see [1, 2, 7, 8, 13, 26, 31, 32] and references therein.

   We now show that when two sides $p,q$ of a 3-way transportation problem are fixed and one side $n$ is variable, the problem is an n-fold integer programming problem, and we could therefore conclude that the convex line-sum 3-way integer transportation problem is solvable in polynomial time. Consider the n-fold programming equations as described after Theorem 1.2 in the introduction. Re-index the arrays as $x = (x^1,\ldots,x^n)$ with each $x^k := (x_{i,j}^k) := (x_{1,1,k},\ldots,x_{p,q,k})$ suitably indexed as a $pq$ vector representing the $k$-th layer of $x$. Let $r := t := pq$ and $s := p + q$, and let $A$ be the $(r+s) \times t$ matrix with $A_1 := I_{pq}$ the $pq \times pq$ identity and with $A_2$ the $(p + q) \times pq$ matrix of equations of the usual 2-way transportation problem for $p \times q$ arrays. Finally, define the right-hand side $b =$

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$(b^0, b^1, \ldots, b^n)$ from the given line-sums by $b^0 := (u_{i,j})$ and $b^k := ((v_{i,k}),(z_{j,k}))$ for $k = 1, \ldots, n$. Then the equations $A_1(\sum_{k=1}^n x^k) = b^0$ represent the constraints $x_{i,j,+} = u_{i,j}$ of all margins with support $\{1,2\}$, where summation over layers occurs, whereas the equations $A_2 x^k = b^k$ for $k = 1, \ldots, n$ represent the constraints $x_{i,+,k} = v_{i,k}$ and $x_{+,j,k} = z_{j,k}$ of all margins with support $\{1,3\}$ or $\{2,3\}$, where summations are within a single layer at a time. Thus, generalizing the recent results of [12] for linear objective functions, we obtain the following remarkable corollary of Theorem 1.2.

**Corollary 1.3** *For any fixed $d, p, q$ there is a polynomial oracle-time algorithm that, given $n$, arrays $w_1, \ldots, w_d \in \mathbb{Z}^{p \times q \times n}$, $u \in \mathbb{Z}^{p \times q}$, $v \in \mathbb{Z}^{p \times n}$, $z \in \mathbb{Z}^{q \times n}$, and convex $c : \mathbb{R}^d \longrightarrow \mathbb{R}$ presented by comparison oracle, solves the convex integer 3-way transportation problem*

$$\max\{\, c(w_1 x, \ldots, w_d x) \ : \ x \in \mathbb{N}^{p \times q \times n}, \ \sum_i x_{i,j,k} = z_{j,k}, \ \sum_j x_{i,j,k} = v_{i,k}, \ \sum_k x_{i,j,k} = u_{i,j} \,\} \ .$$

As mentioned before, this is in contrast with the case when the dimensions of two sides of the tables are variable, in which even the linear integer 3-way transportation problem is NP-hard, see [9, 10, 11].

The following very general extension of Corollary 1.3 holds as well. Consider transportation problems of any fixed dimension $k$ for *long* arrays, namely $m_1 \times \cdots \times m_{k-1} \times n$ arrays where $m_1, \ldots, m_{k-1}$ are fixed and only the length (number of layers) $n$ is variable. Further, let $\mathcal{F}$ be any family of subsets of $\{1, \ldots, k\}$ (the family of supports of fixed margins). Now re-index the arrays as $x = (x^1, \ldots, x^n)$ with each $x^j = (x_{i_1, \ldots, i_{k-1}, j})$ a suitably indexed vector representing the $j$-th layer of $x$. Then this again is a convex n-fold integer programming problem with an $(r+s) \times t$ defining matrix $A$, with $t := \prod m_i$, with $r, s, A_1$ and $A_2$ suitably determined from $\mathcal{F}$, and with the right-hand side determined from the given margins, in such a way that the equations $A_1(\sum_{j=1}^n x^j) = b^0$ represent the constraints of all margins $x_{i_1, \ldots, i_k}$ with $i_k = +$ (where summation over layers occurs), whereas the equations $A_2 x^j = b^j$ for $j = 1, \ldots, n$ represent the constraints of all margins $x_{i_1, \ldots, i_k}$ with $i_k \neq +$ (where summations are within a single layer at a time). We obtain the following corollary of Theorem 1.2 providing the polynomial time solvability of a very broad class of convex integer multiway transportation problems.

**Corollary 3.1** *For any fixed $d, k, m_1, \ldots, m_{k-1}$, and family $\mathcal{F}$ of subsets of $\{1, \ldots, k\}$, there is a polynomial oracle-time algorithm that, given $n$, arrays $w_1, \ldots, w_d \in \mathbb{Z}^{m_1 \times \cdots \times m_{k-1} \times n}$, margin values $u_{i_1, \ldots, i_k}$ for all tuples $(i_1, \ldots, i_k)$ with support in $\mathcal{F}$, and convex $c : \mathbb{R}^d \longrightarrow \mathbb{R}$ presented by comparison oracle, solves the corresponding convex integer multiway transportation problem*

$$\max\{\, c(w_1 x, \ldots, w_d x) \ : \ x \in \mathbb{N}^{m_1 \times \cdots \times m_{k-1} \times n}, \ x_{i_1, \ldots, i_k} = u_{i_1, \ldots, i_k}, \ \mathrm{supp}(i_1, \ldots, i_k) \in \mathcal{F} \,\} \ .$$

## 3.2 Packing problems

We consider the following rather general packing problem, which concerns maximum utility packing of many items of several types in various bins subject to weight constraints. More precisely, the data

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is as follows. There are $t$ types of items. The weight of each item of type $j$ is $v_j$ and there are $n_j$ items of type $j$ to be packed. There are $n$ bins, where bin $k$ has maximum weight capacity $u_k$. In the linear version of the problem, there is one utility matrix $w \in \mathbb{Z}^{t \times n}$ where $w_{j,k}$ is the utility of packing one item of type $j$ in bin $k$, and the objective is to find a feasible packing of maximum total utility. In the more general convex version, there are $d$ utility matrices $w_1, \ldots, w_d \in \mathbb{Z}^{t \times n}$, representing the packing utilities under $d$ different criteria. The total utility is the "balancing" of these linear utilities under a given convex functional $c$ on $\mathbb{R}^d$. By incrementing the number $t$ of types by 1 and suitably augmenting the data, we may assume that the last type $t$ represents "slack items" which occupy the unused capacity in each bin, where the weight of each slack item is 1, the utility under each of the $d$ criteria of packing any slack item in any bin is 0, and the number of slack bins is the total residual weight capacity $n_t := \sum_{k=1}^{n} u_k - \sum_{j=1}^{t-1} n_j v_j$. Let $x \in \mathbb{N}^{t \times n}$ be a variable matrix where $x_{j,k}$ represents the number of items of type $j$ to be packed in bin $k$. Then the convex packing problem is:

$$\max\{\, c(w_1 x, \ldots, w_d x) \ : \ x \in \mathbb{N}^{t \times n}, \ \sum_j v_j x_{j,k} = u_k, \ \sum_k x_{j,k} = n_j \,\} \ .$$

By suitably arranging the variables in a vector, it is not hard to see that this is a convex n-fold integer programming problem with a $(t+1) \times t$ defining matrix $A$, where $A_1 := I_t$ is the $t \times t$ identity matrix and $A_2 := (v_1, \ldots, v_t)$ is a $1 \times t$ matrix. Thus, we obtain the following corollary to Theorem 1.2.

**Corollary 3.2** *For any fixed number $t$ of types and type weights $v_1, \ldots, v_t$, there is a polynomial oracle-time algorithm that, given $n$, item numbers $n_j$, bin capacities $u_k$, utilities $w_1, \ldots, w_d \in \mathbb{Z}^{t \times n}$, and convex $c : \mathbb{R}^d \longrightarrow \mathbb{R}$ presented by comparison oracle, solves the convex integer bin packing problem.*

Note that an interesting special case of bin packing is the classical *cutting stock* problem, and a similar corollary regarding the solvability of a suitable convex cutting stock problem can be obtained as well.

## 3.3 Vector partitioning and clustering

The vector partition problem concerns the partitioning of $n$ items among $p$ players to maximize social value subject to constraints on the number of items each player can receive. More precisely, the data is as follows. With each item $i$ is associated a vector $v_i \in \mathbb{Z}^k$ representing its utility under $k$ criteria. The utility of player $h$ under partition $\pi = (\pi_1, \ldots, \pi_p)$ of the set of items $\{1, \ldots, n\}$ is the sum $v_h^{\pi} := \sum_{i \in \pi_h} v_i$ of utility vectors of items assigned to $h$ under $\pi$. The social value of $\pi$ is the balancing $c(v_{1,1}^{\pi}, \ldots, v_{1,k}^{\pi}, \ldots, v_{p,1}^{\pi}, \ldots, v_{p,k}^{\pi})$ of the player utilities, where $c$ is a convex functional on $\mathbb{R}^{pk}$. In the constrained version, the number $|\pi_h|$ of items that player $h$ gets is required to be a given number $\lambda_h$ (so $\sum \lambda_h = n$). In the unconstrained version, there is no restriction on the number of items per player.

Vector partition problems have applications in diverse fields such as clustering, inventory, reliability, and more - see [3, 6, 14, 19, 20, 24, 25] and references therein. Here is a typical example.

**Example 3.3 Minimal variance clustering.** This is the following problem, which has numerous applications in the analysis of statistical data: given $n$ observed points $v_1, \ldots, v_n$ in $k$-space, group the points into $p$ clusters $\pi_1, \ldots, \pi_p$ so as to minimize the sum of cluster variances given by

$$\sum_{h=1}^{p} \frac{1}{|\pi_h|} \sum_{i \in \pi_h} ||v_i - (\frac{1}{|\pi_h|} \sum_{i \in \pi_h} v_i)||^2 .$$

Consider the instance where there are $n = pm$ points and the desired clustering is balanced, that is, the clusters should have equal size $m$. Suitable manipulation of the sum of variances shows that the problem is equivalent to a constrained partition problem, where $\lambda_h = m$ for all $h$, and where the convex functional $c : \mathbb{R}^{pk} \longrightarrow \mathbb{R}$ (to be maximized) is the Euclidean norm squared, given by

$$c(z) \;=\; ||z||^2 \;=\; \sum_{h=1}^{p} \sum_{i=1}^{k} |z_{h,i}|^2 .$$

If either the number of criteria $k$ or the number of players $p$ is variable, the partition problem is intractable since it instantly captures NP-hard problems [19]. When both $k, p$ are fixed, both the constrained and unconstrained versions of the vector partition problem are polynomial time solvable [19, 24]. We now demonstrate how to get this result as a corollary of Theorem 1.2 by showing that both versions are special convex n-fold integer programming problems. There is an obvious one-to-one correspondence between partitions and matrices $x \in \{0,1\}^{p \times n}$ with all column-sums equal to one, where partition $\pi$ corresponds to the matrix $x$ with $x_{h,i} = 1$ if $i \in \pi_h$ and $x_{h,i} = 0$ otherwise. Let $d := pk$ and define $d$ matrices $w_{h,j} \in \mathbb{Z}^{p \times n}$ by setting $(w_{h,j})_{h,i} := v_{i,j}$ for all $h = 1, \ldots, p$, $i = 1, \ldots, n$ and $j = 1, \ldots, k$, and setting all other entries to zero. Then for any partition $\pi$ and its corresponding matrix $x$ we have $v_{h,j}^\pi = w_{h,j}x$ for all $h = 1, \ldots, p$ and $j = 1, \ldots, k$. Therefore, we obtain that the unconstrained vector partition problem is the convex integer programming problem

$$\max\{\, c(w_{1,1}x, \ldots, w_{p,k}x) \;:\; x \in \mathbb{N}^{p \times n}, \; \sum_h x_{h,i} = 1 \,\} .$$

Suitably arranging the variables in a vector, it is not hard to see that this is a convex n-fold integer programming problem with a $(0 + 1) \times p$ defining matrix $A$, where $A_1$ is empty and $A_2 := (1, \ldots, 1)$. Similarly, the constrained vector partition problem is the convex integer programming problem

$$\max\{\, c(w_{1,1}x, \ldots, w_{p,k}x) \;:\; x \in \mathbb{N}^{p \times n}, \; \sum_h x_{h,i} = 1, \; \sum_i x_{h,i} = \lambda_h \,\} .$$

Again, it can be seen that this is a convex n-fold integer programming problem, now with a $(p+1) \times p$ defining matrix $A$, where now $A_1 := I_p$ is the $p \times p$ identity matrix, and $A_2 := (1, \ldots, 1)$ as before.

Thus, we obtain the following corollary to Theorem 1.2.

**Corollary 3.4** *For any fixed number $p$ of players and number $k$ of criteria, there is a polynomial oracle-time algorithm that, given $n$, item vectors $v_i \in \mathbb{Z}^k$, positive integers $\lambda_h$, and convex $c : \mathbb{R}^{pk} \longrightarrow \mathbb{R}$ presented by comparison oracle, solves the constrained and the unconstrained vector partition problems.*

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