De Loera Version 2001

5. Convex Polytopes

Definition 5.1: A convex polytope is a set that is the convex hull of a finite set of points in E^n. (From here on we shall usually say "polytope" rather than "convex polytope". Sometimes we shall say "d-polytope" when we wish to indicate that it has dimension d.) It seems obvious that a polytope has finitely many extreme points, but this does need proof.

Theorem 5.1: If U is a finite set in E^n, and if $P = \text{con } U$, then the extreme points of P are points in U.

Proof: See Exercise 6, Ch. 3.

Polytopes in E^1 are just the closed segments. The 2-polytopes are the convex polygons. In E^3 their structure becomes more complex. A few 3-polytopes are familiar to most who have had high school geometry - The tetrahedron, octahedron, cube, icosahedron, and dodecahedron. As can be seen by the definition, however, these are but a few of the infinitely many different kinds of 3-polytopes.

There is an alternate way to define a polytope.

Theorem 5.2: If P is a bounded d-dimensional set in E^n, $n \geq 1$, that is the intersection of a finite collection of closed halfspaces, then P is a polytope.

Proof: Our proof is by induction on the dimension of P. If the dimension of P is 0 or 1, the theorem is clearly true.

Let P be $\cap H_i^+$ where each H_i^+ is a closed halfspace bounded by a hyperplane H_i, for $1 \leq i \leq k$. Let p be an arbitrary extreme point of P. Since p is in the intersection of the halfspaces, it is either in the interior of all of the halfspaces or is on one of the bounding hyperplanes. If it is in the interior of each halfspace then it is in the interior of P and is not an extreme point, thus p is on a hyperplane H_i. Now we consider the set

$$P' = H_i \cap (\cap H_j^+) \text{ where } H_j \neq H_i.$$

The intersection of the hyperplane H_i with any one of the closed halfspaces H_j^+ is a closed halfspace in H_i, thus P' is a bounded set that is the intersection of finitely many closed halfspaces in H_i. By induction, P' is a polytope. We also have that p is in P'. Since p is an extreme point of P it is an extreme point of P'. By Definition 5.1, P' is a convex hull of a finite set, and by Theorem 5.1 the extreme points of P' are a subset of this finite set, thus P' has finitely many extreme points.

We now have that any extreme points of P must lie in one of finitely many finite sets of extreme points of sets of the form $H_i \cap (\cap H_j^+)$, thus there are only finitely many extreme points of P. Finally we recall from Theorem 3.6 that a convex body is the convex hull of its extreme points, and this gives us that P is the convex hull of a finite set and is thus a polytope. ■

The converse we shall present without proof:

Theorem 5.2: If P is a d-polytope in E^d then P is the intersection of a finite collection of closed halfspaces.

From these basic properties of polytopes the following can be proved.

Theorem 5.3: The intersection of a d-polytope P with a hyperplane is a polytope. If the hyperplane passes through an interior point of P then the intersection is a (d-1)-polytope.

Theorem 5.4: Every projection of a polytope is a polytope.

Definition 5.2: A face F of a d-polytope P is the intersection of P with a supporting hyperplane. If F is of dimension d-1, then it is called a facet of P. If it is of dimension d-2 it is called a subfacet of P. If it is of dimension 1, it is called an edge and if it is of dimension 0 it is called a vertex.

For example, for 3-polytopes the facet are polygons and the subfacets are edges. We present a list of theorems that give most of the basic properties of faces. We shall omit the proofs as some of them tend to get rather technical.

Theorem 5.5: Every face of a polytope is a polytope.

Theorem 5.6: If F is a face of P and G is a face of F, then G is a face of P.

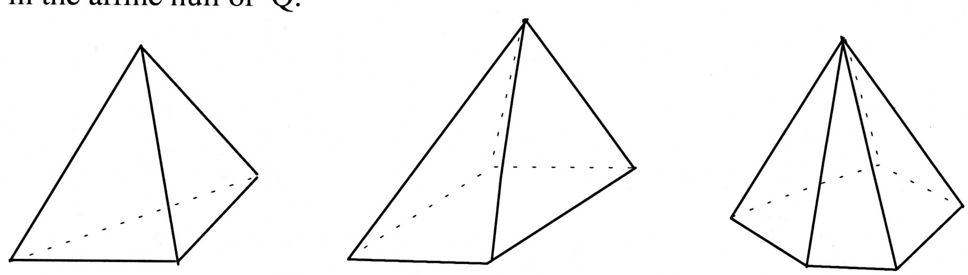
Theorem 5.7: Every point on the boundary of a polytope P lies in a facet of P.

Theorem 5.8: Each subfacet of a polytope P lies in exactly two facets of P.

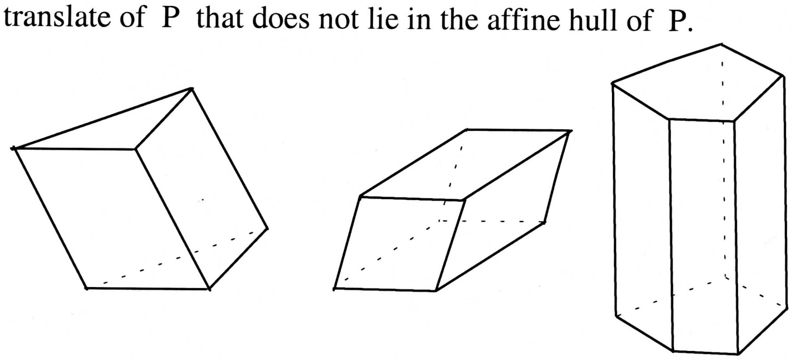
Theorem 5.9: Each vertex of a d-polytope P lies in at least d facets of P and at least d edges of P.

We shall now give some examples of polytopes.

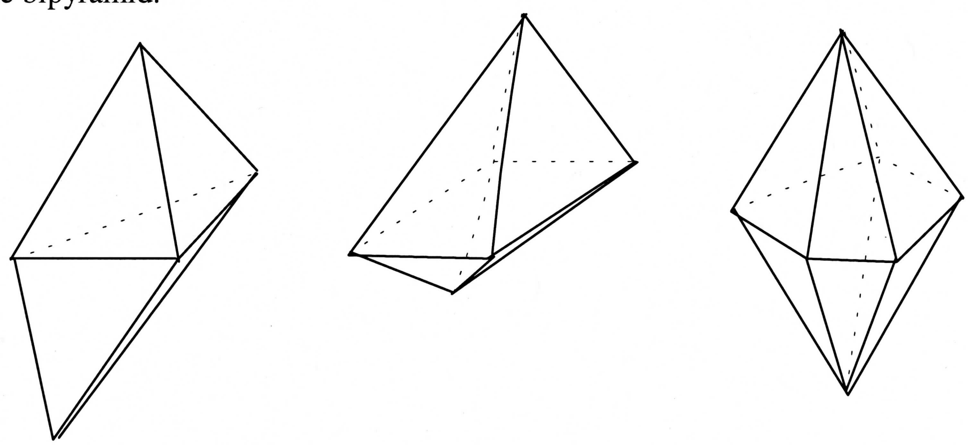
Definition 5.3: A d-pyramid is the convex hull of a (d-1)-polytope Q and a point not in the affine hull of Q.



Definition 5.4: A d-prism is the convex hull of two (d-1)- polytopes P and P' where P' is a translate of P that does not lie in the affine hull of P.



Definition 5.5: A d-bipyramid is the convex hull of a (d-1)-polytope P and a segment that intersects the interior of P, with one end point on one side of the affine hull of P (in E^d) and the other end point on the other side. The polytope P is called the equator of the bipyramid.



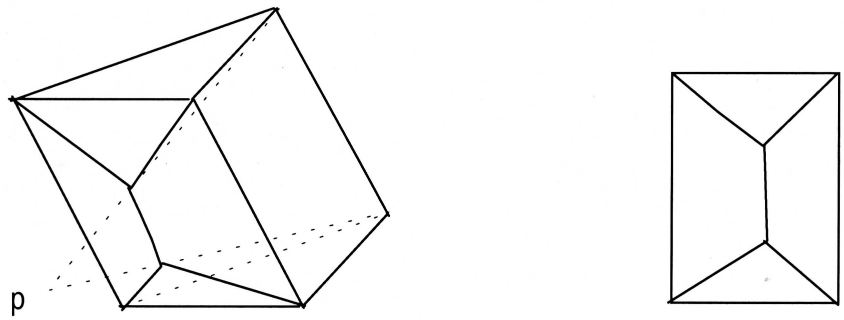
Definition 5.6: A 0-simplex is a point. A d-simplex is a pyramid over a (d-1)simplex.

Definition 5.7: A 0-cube is a point. A d-cube is a prism over a (d-1)-cube.

Definition 5.8: A 1-octahedron is a segment. A d-octahedron is a bipyramid with a (d-1)-octahedron as its equator.

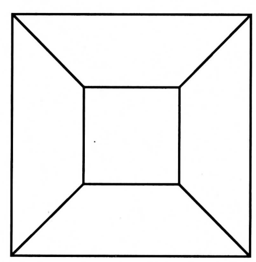
All but the simplest of the 3-polytopes are difficult to draw, and 4-polytopes are impossible to draw. There is, however a simple method of showing the facial structure of 3- and 4-polytopes. It involves the use of Schlegel diagrams.

Definition5.9: Let P be a d-polytope, and let p be a point not in P but close to the centroid of a facet F of P. If we project the boundary of P onto F by projecting through the point p, the image of the boundary in F is called a Schlegel diagram of P.



A Schlegel diagram of a triangular prism

The Schlegel diagram shows the combinatorial structure of the polytope, in other words it shows how the various faces "fit together". The Schlegel diagram does not accurately represent any metric properties of the polytope, that is, it doesn't show properties that are measured, such as area, volume, edge length, angle size, etc. So, for example from the Schlegel diagram of the cube



we see that there are eight vertices twelve edges and six facets. We see that every facet is a quadrilateral and that each vertex has exactly three edges meeting it. We cannot, however, say what the volume of the original cube was.

The Schlegel Diagrams of 3-polytopes are examples of a large class of structures called graphs.

Definition 5.10: A graph is a configuration consisting of a finite set of points (called the vertices of the graph) with various pairs of of the points joined by arcs (called the edges of the graph). The configuration consisting of just the vertices and edges of a polytope (or of its Schlegel diagram) is often called the graph of the polytope.

DEF. 5.10: A GRAPH is a configuration consisting of finite set of vertices, with a set pairs of vertices joined by Edges. Every polyhedron has a graph from its 0-faces (vertices) and the 1-faces (edges) connecting them.

Definition 5.11: If v is a vertex of a graph or of a polytope, the number of edges meeting v is called the valence of v. If v has valence i we say that v is i-valent.

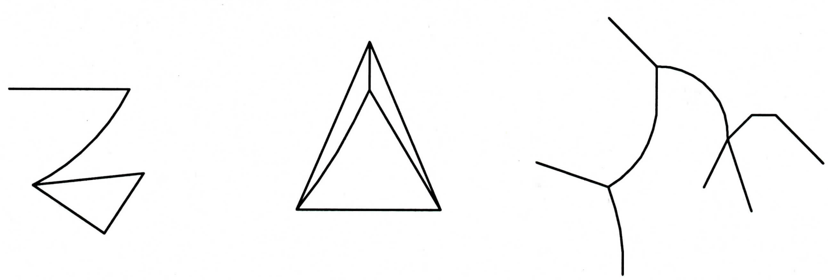
We shall prove one of the most important properties of 3-polytopes by proving a similar theorem about a certain class of graphs.

Definition 5.12: A graph is planar provided it can be drawn in the plane without any two edges crossing each other.

Definition 5.13: A graph is connected iff between any two vertices there is a path of edges joining them.

Definition 5.14: If a graph is drawn in the plane (without edges crossing) then the graph breaks the plane into regions. These regions are called the faces of the graph.

For example the first graph below has two faces (one is an unbounded region) the second has four and the third has just one.



The following is a fundamental theorem in graph theory and will establish an important combinatorial property of 3-polytopes.

Theorem 5.10: If G is a planar connected graph drawn in the plane, and if V, E, and F are the numbers of vertices edges and faces, respectively, of G then V-E+F = 2.

Proof: The proof is by induction on the number of edges of G. If G has no edges, then since it is connected it must have exactly one vertex, thus V-E+F = 1-0+1 = 2.

MENTION: Steinitz theorem and Balinski's theorem

Suppose that the theorem is true for planar connected graphs with k edges and suppose that G has k+1 edges.

We are going to remove an edge from G and apply induction. We must be careful, however, because removing an edge could leave the remaining graph disconnected and the induction hypothesis would not apply.

Suppose that we start at a vertex and begin traveling along edges, never backtracking along an edge. Since there are a finite number of edges, one of two things will happen. Either we will arrive at a vertex from which we can't leave (and thus the vertex meets just one edge of the graph) or we return to some vertex that we have visited before, in which case part of our journey is a simple closed curve.

If we arrive at a vertex v that meets just one edge e of the graph, we remove v and e producing a graph G'. Clearly the remaining graph is still connected. By induction, $V-E+F = 2$ for the graph G'. But G has one more vertex and one more edge than G' and thus going from G' to G gives a net change of 0 in $V-E+F$.

If our journey has a simple closed curve C in it, we remove an edge e of C to produce G'. The graph G' is still connected because given any two vertices in G there is a path connecting them. If this path used e we can substitute a path along the remainder of C in place of e. By induction $V-E+F = 2$ for G'. Now G' has one less face than G because a face inside C was merged with a face outside C when e was removed. Thus G has one more face and one more edge than G'. The net change for $V-E+F$ is again 0. ■

Theorem 5.11(Euler's equation): <u>If V, E, and F are the numbers of vertices, edges, and facets, respectively of a 3-polytope P, then V-E+F = 2</u>.

Proof: The theorem follows immediately from the previous theorem since the Schlegel diagram of P is a planar connected graph. ■

The number of vertices of a 3-polytope does not uniquely determine the number of facets as is shown by the pyramid over a rectangel and a bypyramid with a triangular equator. In face no one of V, E, or F uniquely determines either of the other two. This leads to questions about the maximum and minimum numbers of one given the number of another.

Lemma 5.1: <u>For any 3-polytope P, $2E \geq 3V$ and $2E \geq 3F$</u>.

Proof: Imagine that we place a mark on each edge near each vertex. The number of marks will then be 2E because we have placed two on each edge, one near each of its two vertices. Since each vertex meets at least three edges, the number of marks is at least 3V, thus $2E \geq 3V$. A similar marking argument yields the second inequality (See Exercise). ■

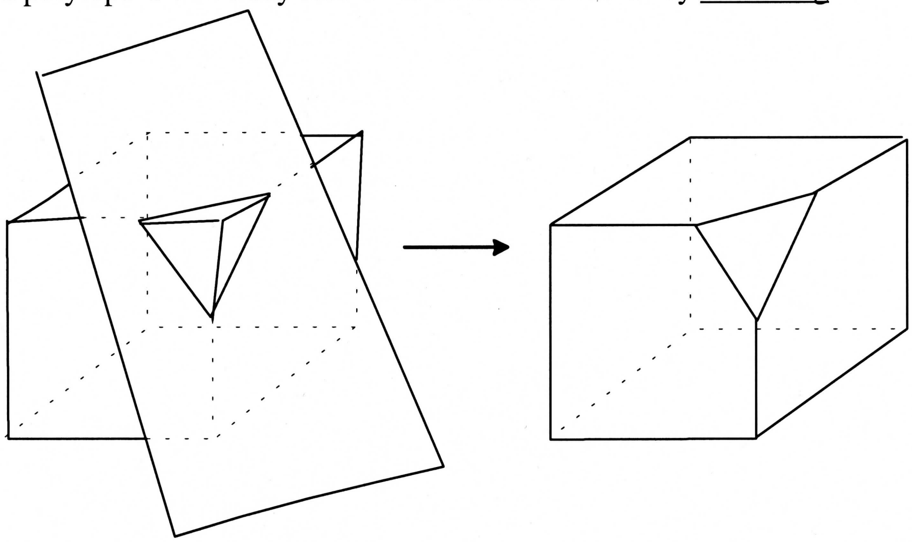
Theorem 5.12: <u>For any 3-polytope $F \geq (v+4)/2$</u>.

Proof: By Lemma 5.1 we have $E \geq 3V/2$. Substituting for E in Euler's equation gives the inequality. ■

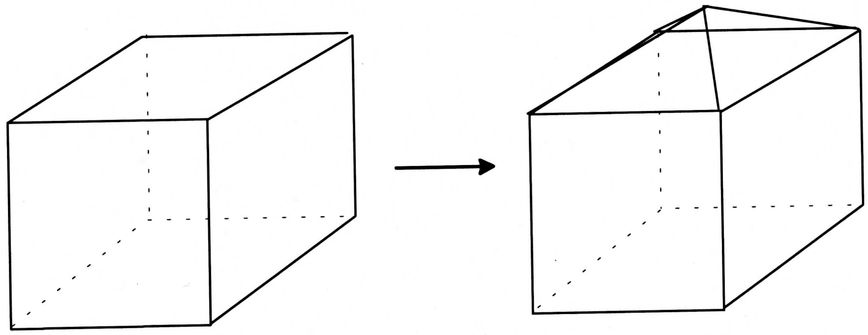
Theorem 5.12 only answers half the question about the minimum number of facets for a given number of vertices. There is still the question of whether the inequality is sharp. That is, for a given V is there a 3-polytope with (V+4)/2 facets. Unless we phrase this question more carefully, the obvious answer is no. There is no such 3-polytope for V = 9, for example. (Do you see the simple reason why?)

We shall introduce some more construction techniques sos that we will have a much richer family of examples that we can draw from.

Definition 5.15: Suppose v is a vertex of a d-polytope P, suppose that H is a hyperplane strictly separating v from the other vertices of P, and suppose that H^+ is the halfspace of H containing the other vertices of P. The intersection of P with H^+ is a polytope P'. We say that P' is obtained from P by truncating v.



Definition 5.16: Let F be a facet of a polytope P and let v be a point very close to the centroid of F. If v is close enough to F then taking the convex hull of v and P consists of erecting a pyramid with F as a base and v as the apex and "gluing" the pyramid to P. We say that the polytope P' = (con P∪{v}) is obtained from P by capping the facet F.



Theorem 5.13: For every even integer $v \geq 4$, there exists a 3-polytope with v vertices and $(v+4)/2$ facets.

Proof: The proof of this theorem is a good example of the technique of strengthening the induction hypothesis to make induction work. We shall see that we may more easily prove the following statement:

For every even integer $v \geq 4$, there is a 3-polytope with $(v+4)/2$ facets and a 3-valent vertex.

Our proof is by induction on n. For $n = 4$ the tetrahedron is an example (and the only example) with four facets and four 3-valent vertices. Suppose that the theorem is true for $n = k$. Since we are proving this for even n, our induction step is to show that the theorem holds for $k+2$. Let P be a 3-polytope with k vertices, $(k+4)/2$ facets and a 3-valent vertex v. Let P' be obtained from P by truncation vertex v. Then P' has $k + 2$ vertices and $(k+6)/2$ facets. Furthermore, each of the three vertices created by truncating is 3-valent. ■

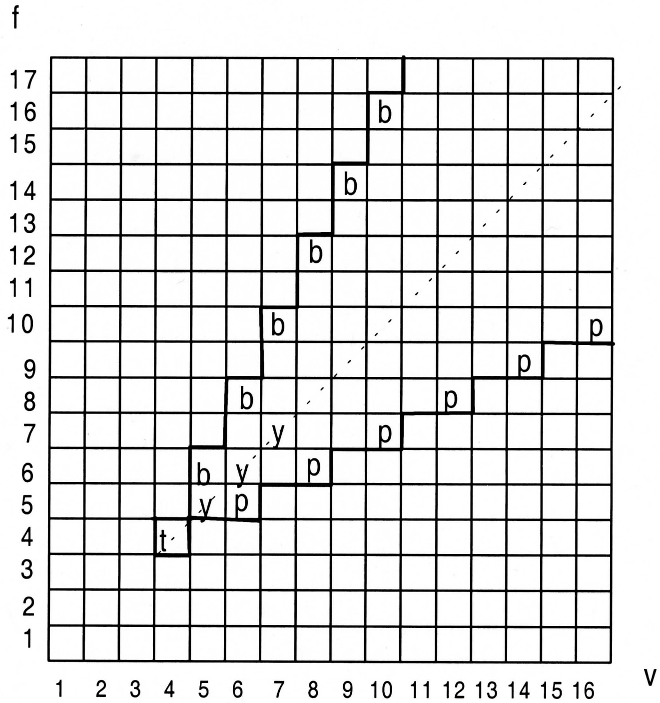
Theorem 5.14: For every even $v \geq 4$, there is a 3-polytope with v vertices and $2v - 4$ facets. (See Exercise)

We are now ready to look at a more general question: Given positive integers v, e, and f, when will there exist a 3-polytope with v vertices, e edges, and f facets? We have seen that the three integers must satisfy Euler's equation, thus if we have two of these numbers the third is uniquely determined. It therefore suffices to ask the question: Given positive integers v and f, when are there exist polytopes with v vertices and f facets?

We have seen two inequalities that v and f must satisfy. We shall now show that as long as v and f satisfy these two inequalities the polytopes will exist.

The following figure shows the first quadrant of the vf-plane. The region that satisfies our two inequalities is outlined. We shall show that there is a polytope corresponding to each square in the region. The square corresponding to the tetrahedron has a "t" in it. We have placed "p's" and "b's" in the squares that correspond to prisms and bipyramids, respectively. Several squares corresponding to pyramids have been filled with "y's", and we note that all squares on the diagonal will correspond to pyramids.

Now, note that if polytope P occupies a square and if P' is obtained from P by truncating a 3-valent vertex, then P' occupies a square one row up and two columns over from P. If P' is obtained from P by capping a triangular face then P' occupies a square two rows up and one column over from P.



Next note that every unfilled square below the diagonal is one row up and two columns over from a square in the region that is on or below the diagonal. Every polytope indicated in the figure that is on or below the diagonal has a 3-valent vertex, furthermore truncating a 3-valent vertex creates a new polytope with a 3-valent vertex.

Every unfilled square above the diagonal is one column over and two rows up from a square in the region that is on or above the diagonal. All polytopes indicated in the figure that are on or above the diagonal have triangular facets, and capping a facet creates a new polytope with a triangular facet.

Thus, with these two operations we can fill out the rest of the region, and we have:

Theorem 5.15: <u>For every v and f greater than or equal to 4, there is a 3-polytope with v vertices and f facets provided 2v-4 ≤ f ≤ (v+4)/2</u>.

Exercises

1. Suppose that every vertex of a 3-polytope P is 4-valent. Find an equation for v in terms of e.

2. Suppose that each facet of a 3-polytope P has at least four sides. Find a sharp inequality for f in terms of e.

3. Describe an infinite family of 3-polytopes all of whose facets are 4-sided.

4. Can a 3-polytope have all vertices 4-valent and all facets 4-sided ?

5. Prove that no 3-polytope has exactly seven edges.

6. Prove that for any $n \geq 6$ and $n \neq 7$ there exists a 3-polytope with exactly n edges.

7. Prove Theorem 5.3.

8. Prove Theorem 5.8.

9. Prove theorem 5.9.

10. Prove Theorem 5.10.

11. The triangular prism has two types of Schlegel diagrams. One type is shown in this chapter. Draw the other type.

12. Draw a schlegel diagram for a 3-dimensional octahedron.

13. Draw a picture of the Schlegel diagram of a bypyramid over a tetrahedron.

14. Draw the Schlegel diagram of a the polytope obtained by truncating each vertex of a tetrahedron.

15. There are simpler examples than given in the text to show that the inequalities $F \leq (V+4)/2$, and $F \geq 2V-4$ are sharp. What are they?

16. Let p_i be the number of i-sided facets of a 3-polytope P. For example, for the cube $p_3 = 0$, $p_4 = 6$, and $p_i = 0$ for $i \geq 5$. For a triangular prism $p_3 = 2$, $p_4 = 3$, and $p_i = 0$ for $i \geq 5$. Consider a Schlegel diagram for P. This diagram consists of f-1 polygons (called the <u>bounded faces</u> of the diagram) filling out a polygon F (The polygon F is the facet that the boundary was projected into, and is called the <u>unbounded face</u> of the diagram.) Suppose we take a bounded face and take the sum of the angles of that face. In terms of the number of edges of that face, what sum do we get? Suppose we do this for each bounded face and add the results. In terms of the p_i's what sum do we get?

Suppose now that instead of adding these angles one face at a time, we add the angles by taking a sum at each vertex and adding the results. What sum do we get? Obtain Euler's equation by equating the two sums obtained by adding the angles in these two different ways.

The following useful facts can be established using linear algebra, and are useful in some of the above problems. You may use them here without proof.

a) If a hyperplane H contains a relative interior point of a k-polytope P in E^n and does not contain P then $P \cap H$ has dimension k-1.

b) If a hyperplane H contains a relative interior point p of a k-face, $k \leq d-1$, F of a polytope P, and if H contains F, then an arbitrarily small movement of H can be made so that the resulting hyperplane contains p and does not contain F.

c) If a hyperplane H intersects a k-polytope P in E^n and misses its vertices, then H intersects a relative interior point of P.

6. Eberhard's Theorem

In this chapter we shall look at problems such as these:

1. Is there a 3-polytope with three triangles, one quadrilateral and one hexagon (as its only facets)?

2. is there a 3-polytope with five pentagons eight hexagons and 17 7-gons?

3. Is there a 3-polytope with 127 triangles 44 quadrilaterals and 11 octagons?

The first question is easily answered without any new theorems (see Exercise 1). After proving our main theorem of this chapter we shall see that the answer to the second question is no. As for the third question, I don't know the answer.

The general question would be the following: Let a set of numbers p_i be given. Is there a 3-polytope with p_i i-sided facets? For example the third question would be: Is there a 3-polyotpe for which $p_3 = 127$, $p_4 = 44$, and $p_8 = 11$ (and, of course, $p_i = 0$ for all other i)?

Although this is similar to a question that we answered in Chapter 5, a general method for answering this question has never been found. We shall examine partial solutions.

We begin with the most basic inequality known for the numbers p_i.

Definition 6.1: Let p_i be the number if i-sided facets of a 3-polytope P. then the vector $(p_3,p_4,...,p_i,...)$ is called the p-vector of P.

For example the p-vector for a pentagonal prism is (0,5,2). The p-vector for a pyramid over a 9-gon would be (9,0,0,0,0,0,1)

The above question is the same as asking: Given a vector, when is it the p-vector of some 3-polytope?

Definition 6.2: A 3-polytope is simple iff each vertex has valence 3.

The tetrahedron, cube, and prisms are examples of simple 3-polytopes.

Theorem 6.1: If $(p_3, \ldots, p_n)$ is the p-vector of a 3-polytope, then $\Sigma\,(6-i)p_i \geq 12$, with equality iff the polytope is simple.

Proof: Let V, E, and F be the numbers of vertices, edges and facets, respectively, for P. Suppose in each facet we place a mark near the middle of each edge. Then the number of marks will be Σip_i. Since two marks were placed near each edge, the sum is also 2E.

The sum $\Sigma 6p_i$ will be six times the number of facets, thus $\Sigma(6-i)p_i = 6F - 2E$.

Now recall from Chapter 5 that we always have $2E \geq 3V$, and observe that in our derivation of that inequality, we actually had equality if each vertex had valence three. Recall also Euler's equation $V-E+F = 2$, from which we get $6V-6E+6F = 12$. now if we substitute 4E for 6V, the inequality $2E \geq 3V$ tells us that we will get $6F - 2E \geq 12$, with equality if the polytope is simple, thus we have:

$$\Sigma(6-i)p_i \geq 12, \quad \text{with equality if the polytope is simple.} \blacksquare$$

We can now see that the answer to question 2 is no, because plugging in to our inequality we would get $1{\cdot}5 + 0{\cdot}8 - 1{\cdot}17 \geq 12$, which is not true, thus there could not be such a 3-polytope.

Corollary 6.1: Every 3-polytope has a facet with fewer than six edges.

Corollary 6.2: There is no polytope all of whose facets are hexagons.

Suppose we use our inequality to tackle question 3. Plugging in we get

$3\cdot127 + 1\cdot44 - 2\cdot11 \geq 12$, which is a true statement. Unfortunately, this tells us nothing. Theorem 6.1 does not tell us that a polytope will exist when the inequality is satisfied, in fact such a polytope won't always exist (See Exercise 8). There are numerous results covering special cases, the most famous is Eberhards's Theorem (Incredibly, Eberhard really was the one who discovered and proved it!).

Theorem 6.2: (Eberhard's Theorem) If $\Sigma(6-i)p_i = 12$, then there exists a value for p_6 such that $(p_3, ..., p_n)$ is the p-vector of some 3-polytope.

Note that the coefficient of p_6 is 0 in the sum, thus the sum is not effected by the number of hexagons. Eberhard's Theorem tells us that if someone specifies how many facets there are of all sizes except 6, and if these numbers satisfy the equation, then one can find a number of hexagons so that the polytope exists.

We regret being unable to furnish the proof of Eberhard's Theorem here. The original proof filled an entire book. In the 1960's Grünbaum, making use of a powerful theorem about Schlegel diagrams, which was unavailable to Eberhard, was able to reduce the proof to about 15 typewritten pages. Suffice it to say that this is a deep result about 3-polytopes.

Exercises

1. Prove without using the inequality developed in this chapter, that (3,1,0,1) is not a p-vector for a 3-polytope.

2a. Let v_i be the number if i-valent vertices of a 3-polytope. Use a marking process to evaluate Σiv_i.

b. What can be said about $\Sigma(6-i)v_i$?

3. A 3-polytope is called simplicial iff each facet is a triangle. Show that the inequality that you obtained in problem 2b is an equation when P is simplicial.

4. Prove that for any 3-polytope, $\Sigma(4\text{-}i)(v_i + p_i) = 8$. Hint: use your knowledge about $\Sigma 4v_i$ and $\Sigma 4p_i$.

5. Does there exist a 3-polytope whose vertices are all 4-valent and whose facets are all quadrilaterals?

6. Does there exist a 3-polytope for which each two facets have a different number of edges?

7. Suppose that every vertex of a 3-polytope is 4-valent. What can be said about $\Sigma(4\text{-}i)p_i$?

8. Find a vector, $(p_3, \ldots, p_n)$ such that $\Sigma(6\text{-}i)p_i = 12$, but there is no 3-polytope with that p-vector.

Chapter 7. Polytopes and Map Coloring

A map is a planar graph (drawn in the plane) such that each vertex has valence at least three and each region is bounded by a simple closed curve.

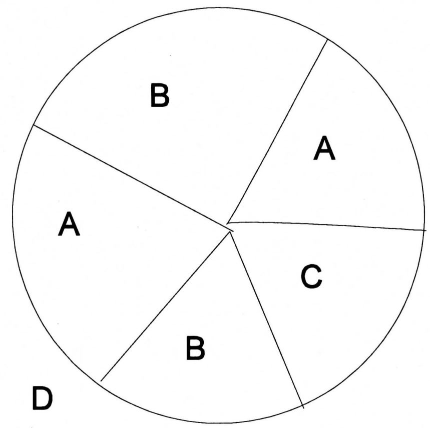
a map **not a map**

Note that Schlegel diagrams of 3-polytopes are all maps.

Definition 7.1: The regions of a map are called countries.

Definition7.2: A map is said to be colored with n colors iff colors are assigned to the countries such that no two countries meeting on an edge have the same color and at most n colors have been used. A map colored with n colors is said to be n-colored.

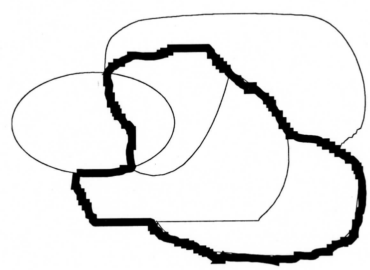
The following is an example of a map that is colored with four colors.



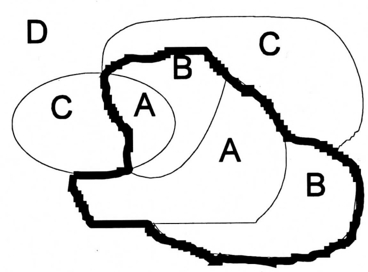
In 1852 it was conjectured that every map could be colored with four or fewer colors. In 1879 an English barrister named A. B. Kempe published a "proof" of the conjecture. It was not until eleven years later that an error in Kempe's work was found. Modifying Kempe's methods, Heawood proved that every map could be colored with five colors. There were, however no maps known that actually required five colors, thus the original conjecture was still unproved. As the years passed and no one could prove it, it became one of the most famous unsolved problems in mathematics. It was not until 1976 that Kenneth Apel and Wolfgang Haken at the University of Illinois finally produced a proof of the conjecture. It has remained a very controversial proof for two reasons. First, one part of the proof depends on generating a set of approximately 2000 maps (a set which we shall call M) according to certain rules. The set M was generated by hand and required nearly two years to accomplish. (To my knowledge no one has ever checked that part of the proof.) Second, another part involves using a computer to verify that the maps in M have certain properties. This was done using over 1000 hours of computer time. This part has been independently verified by others who have used their own programs to check the maps in M, however it has remained a source of discomfort to many mathematicians that an essential part of the proof is the work done by a computer. As for the first part of the proof, Apel and Haken have given a probabilistic argument that shows that if there were any mistakes in generating M, the probability that it will effect the validity of their proof is virtually zero!

In this chapter we shall see an unsuccessful attempt at a proof of the 4-color conjecture,and some of its interesting consequences.

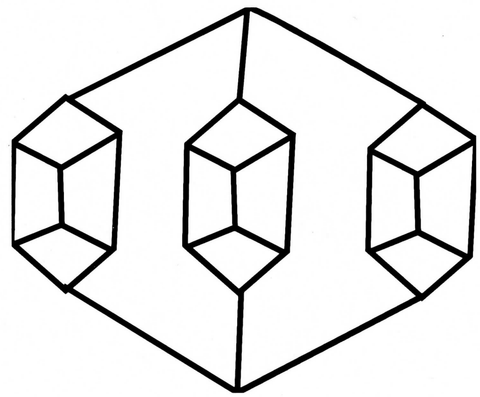
Definition 7.3: A Hamiltonian circuit in a map, or in a polytope, is a simple closed curve, consisting of edges, passing exactly once through each vertex. The following is an example of a Hamiltonian circuit in the above map:



Observe that if we have a map with a Hamiltonian circuit, we can color the countries inside the circuit with colors A and B and the outside countries alternately C and D, and we have 4-colored the map.



This suggests that one could prove the 4-color conjecture by proving that every map has a Hamiltonian circuit. The following map, however, shows that such a statement is not true (See Exercise 1):



It's a rather unusual coincidence that in the early work on the 4-color problem one of the theorems that was proved (actually it is a consequence of several theorems) was the following:

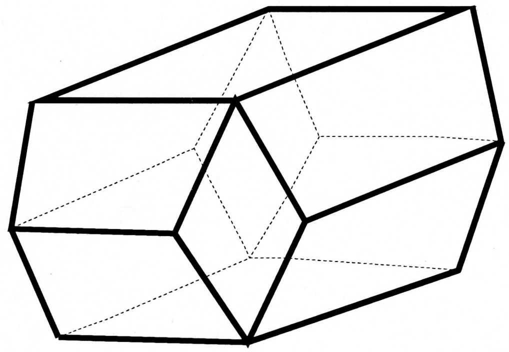
Theorem 7.1: If the four color conjecture is true for Schlegel diagrams of 3-polytopes then it is true for all maps.

This suggests that one could prove the 4-color theorem by proving:

Conjecture 7.1: The Schlegel diagrams of all 3-polytopes have Hamiltonian circuits.

(Or equivalently: Every 3-polytope has a Hamiltonian circuit.)

Unfortunately, this conjecture is not true as the following polytope shows.

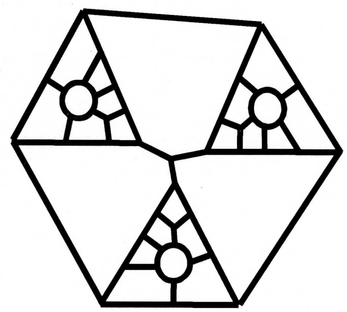


This polytope is called the Rhombic Dodecahedron. Note that each 3-valent vertex is surrounded by 4-valent vertices, and each 4-valent vertex is surrounded by 3-valent vertices, thus any circuit will alternate 3- and 4-valent vertices. But, there are six 4-valent and eight 3-valent vertices, thus no circuit can contain all vertices.

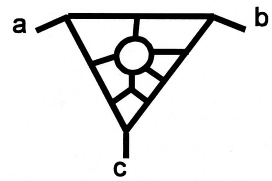
This counter example, however, did not end this line of investigation, for the following had also been proved:

Theorem 7.2: If all the Schlegel diagrams of all simple 3-polytopes are 4-colorable, then all maps are 4-colorable.

The Rhombic Dodecahedron is not a simple polytope, and thus is not a counter example to the conjecture that the simple ones all have Hamiltonian circuits. For years mathematicians searched for a simple 3-polytope with no Hamiltonian circuit, with no success. They also tried to prove that they all had such circuits, also without success. In 1932 a biologist named Jules Chuard published a "proof" that all simple 3-polytopes had Hamiltonian circuits (and thus claimed to have solved the 4-color problem). This proof was shown to be incorrect when in 1946 William Tutte found a simple 3-polytope with no Hamiltonian circuit. The following is a Schlegel diagram of his polytope.



The proof that there is no Hamiltonian circuit uses a property of a part of the graph called a Tutte triangle:



As one can check with a few minutes work exhausting cases, no path can enter and leave this portion of the graph at a and b and pass through every vertex of the Tutte triangle. This means that if there is a Hamiltonian circuit then the portion passing through each Tutte triangle must use the edge labeled c. Since there are three of these edges and they meet at a vertex we have that the circuit uses all three edges at that vertex. This is a contradiction because a circuit uses exactly two edges at a vertex.

Tutte's discovery did not end this way of trying to prove the 4-color conjecture. Note that if we take Tutte's graph and cut three of the edges meeting a Tutte triangle we will have disconnected the graph. In fact we will have disconnected it in a way that there will be two separate pieces each containing a country. When the graph of a simple 3-polytope has the property that we cannot separate two countries by cutting just three edges we say that the graph is cyclically 4-connected. Tutte's graph is not cyclically 4-connected. Examples that are cyclically 4-connected include the graphs of the prisms over polygons of at least four sides, and the dodecahedron. The following is another theorem that was known to researchers working on this problem:

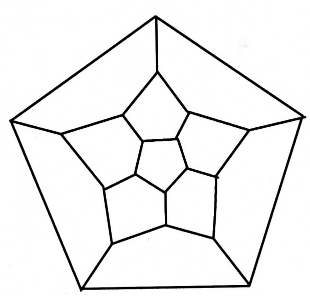
Theorem 7.3: If every cyclically 4-connected simple Schlegel diagram is 4-colorable then every map is 4-colorable.

In 1960 Tutte found a cyclically 4-connected simple Schlegel diagram without a Hamiltonian circuit. HOWEVER....

It was also known that if all cyclically 5-connected simple Schlegel diagrams were4-colorable then all maps were 4-colorable.

Definition 7.4: A simple Schlegel diagram of a 3-polytope is cyclically n-connected iff one can separate two ccountries by cutting edges, but one cannot separate two countries by cutting fewer than n edges.

For example the graph of the cube is not cyclically 5-connected but the graph of the dodecahedron is.



Schlegel diagram of the dodecahedron

Unfortunately, in 1965 Walter found a cyclically 5-connected simple Schlegel diagram with no Hamiltonian circuit. His example was quite complex, having 82 countries and 160 vertices.

This however was not the end of the line. There is something called strong cyclic connectivity.

Definition 7.5: A simple Schlegel diagram of a 3-polytope is <u>strongly cyclically n-connected</u> iff is cyclically n-connected, and the only way to separate two countries by cutting n edges is by cutting the edges meeting an n-sided country.

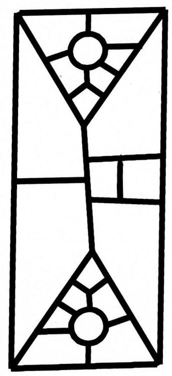
Theorem 7.4: <u>If the strongly cyclically 5-connected simple Schlegel diagrams of 3-polytopes are all 4-colorable then all maps are 4-colorable</u>.

There was still hope. Perhaps all of these had Hamiltonian circuits. This would imply that they were all four colorable and thus all maps would be 4-colorable.

About 1970 Kozyrev, Grinberg and Tutte found a strongly cyclically 5-connected simple Schlegel diagram without a Hamiltonian circuit. To everyone's surprise, it had fewer vertices than Tutte's first example.

This finally ends this line of attack on the 4-color problem because there are no planar cyclically 6-connected graphs (Exercise 10).

Although one can't prove the 4-color conjecture with Hamiltonian circuits, the circuits themselves have become interesting. One problem of interest is to determine the smallest simple 3-polytope with no Hamiltonian circuit (here, smallest, means lest number of vertices.) In the middle 1960's three mathematicians independently found the following non-Hamiltonian example:



This graph has 38 vertices, and no smaller example has ever been found. Since it's discovery mathematicians have been trying to prove that 38 is the minimum number of vertices of a simple 3-polytope with no Hamiltonian circuit. The most recent result is that all simple 3-polytopes with 34 or fewer vertices have Hamiltonian circuits. Since simple 3-polytopes always have an even number of vertices, the question becomes: Is the minimum number of vertices for non-Hamiltonian simple 3-polytopes 36 or 38?

Exercises

1. Prove that the map given at the top of page 60 has no Hamiltonian circuit.

2. If the path does not return to it's starting point but passes through every vertex, it is called a Hamiltonian path. Does the rhombic dodecahedron have a Hamiltonian path? If not, what is the maximum length of any path (that doesn't intersect itself)?

3. Prove that if every map in which every vertex is 3-valent is 4-colorable then every map is 4-colorable. Hint: Suppose you have a map that has vertices of valence greater than three. What happens if you replace such vertices with small countries?

4. Prove that every simple 3-polytope has an even number of vertices.

5.Prove that for every integer n there is a map with at least n countries that is colorable with 3 colors.

6. Prove that for every n there is a map with at least n countries that can be colored with 2 colors.

7. Let P be the polytope obtained by capping each facet of a tetrahedron. Prove that P has a Hamiltonian circuit.

8. Let Q be obtained by capping every facet of an octahedron. Prove that Q does not have a Hamiltonian circuit.

9. If P is a polytope obtained from polytope Q by capping each facet, then P is called the <u>Kleetope</u> over Q (named after the great geometer Victor L. Klee). In problem 8 you showed that the Kleetope over an octahedron has no Hamiltonian circuit. Prove that the Kleetope over any simplicial 3-polytope, other than the tetrahedron, has no Hamiltonian circuit.

10. Prove that there are no planar cyclically 6-connected 3-polytopes. Hint recall that there must be a facet with five or fewer edges.

11. Prove that the 38 vertex example on page 63 has no Hamiltonian circuit. Hint: suppose that there is such a circuit. If the two Tutte triangels are shrunk to vertices what does this do to the circuit? What edges must the resulting circuit use?