Convex Geometry: MATH 114

The following list collects all the problems on which you will be examined from the second half of the course. There will be some computer projects assigned too.

Exercises for second half of course

A *Section 12 Chakerian Sangwine-Yager notes:*

  i Go over problem 1

  ii Use Carathéodory's theorem to show that the convex hull of a compact set is compact.

  iii In $\mathbb{R}^2$ let $x_1 = (1,0), x_2 = (1,3), x_3 = (4,3), x_4 = (4,0)$. Take $a = (7/4, 5/4)$. Show that a is in the convex hull of the x_i and then express a as a convex combination of only three of them.

B *Section 13 Chakerian, Sangwine-Yager notes:*

  (a) Problems 1,2, and 7.

  (b) Learn one proof Helly's theorem (there are two in the notes). Make sure you know how to explain it well to someone.

The following problems are coming from either (A) De Loera notes "Actually doing it" or (B) De Loera "other notes" I will mark that here as (A) or (B), but some problems are not from those sources.

1. From (A) Prove Lemma 2.2.2 that the set $M_{n\times n}(k)$ of all $n \times n$ matrices whose sums along rows, columns, or diagonals are equal to the same constant (*magic squares*) k is a bounded polyhedron.

2. From (A) Prove Lemma 2.2.4 that the dimension of the polyhedron $M_{n\times n}(k)$ of all $n \times n$ magic squares with magic sum is equal to $(n-1)^2$.

3. From (A) Prove Proposition 2.3.2

4. From (A) Prove Proposition 2.3.4

5. From (B) Chapter 5: Write a proof Theorem 5.1 and show that every vertex of a polyhedron is an extreme point.

6. From (B) Chapter 5 Prove Theorem 5.5

7. From (A) Let $P = conv(v_1, v_2, \ldots, v_m)$ and also has an inequality representation $P = \{x : Ax \leq b\}$ with A a $m \times d$ matrix. Prove that $conv(v_i, v_j)$ is a 1-dimensional face of P if the rank of the matrix $A_I(z)$ is $d-2$ where $z = 1/2(v_i + v_j)$.

8. From (A) Let $P = \{x \in R^d : Ax \leq d\}$ be a polyhedron. Prove that if P contains a line $\{v + tu : t \in R\}$ with u non-zero directional vector, then $Au = 0$.

9. From (A) Prove the following version of Farkas' lemma (HINT: use the version we proved in class as a lemma) $\{x : Ax \leq b, x \geq 0\} \neq \emptyset \iff$ When $y^T A \geq 0,$ then $y^T b \geq 0$

10. For each of the following cases either give a 3-polytope having the proposed f-vector or tell why there is no such polytope. $(f_0, f_1, f_2) = (27, 55, 26)$, $(f_0, f_1, f_2) = (12, 23, 13)$,$(f_0, f_1, f_2) = (35, 51, 18)$.

11. From (B) Chapter 5 Suppose that every vertex of a 3-polytope is 4-valent. Find an equation for v (number of vertices) in terms of e (number of edges).

12. From (B) Chapter 5 Describe an infinite family of 3-polytopes all of whose facets are 4-sided polygons.

13. From (B) Chapter 5 Prove that no 3-polytope has exactly 7 edges.

14. From (B) Chapter 5 Prove that for any $n \geq 6$ and $n \neq 7$ there exists a 3-polytope with exactly n edges.

15. From (B) Chapter 5 exercises (page 52) problem 6, 11, 12, 14.

16. From (B) Chapter 6 Let v_i the number of i-valent vertices of a 3-polytope. Estimate $\sum_i i * v_i$. What can be said of the quantity $\sum_i (6-i)v_i$? HINT: You should get inequalities involving faces and/or edge of the polytope.

17. From (B) Chapter 6 A 3-polytope is simplicial iff each facet is a triangle. Show that the inequalities of the previous problem turn into equations.

18. From (B) Chapter 6 Prove that for any 3-polytope $\sum_i (4-i)(v_i + p_i) = 8$ HINT: use your knowledge about $\sum_i 4v_i$ and $\sum_i 4p_i$.

19. From (B) Chapter 6 exercises (page 56) problem 5,6,8.

20. Does there exist a 3-polytope for which each two facets have a different number of edges?

21. Show that if P is a 3-polytope such that each facet is a regular triangle, then P has at most 12 vertices and at most 20 facets. Then prove that there is no such 3-polytopes with 18 facets.

22. If a d-polytope has V vertices, what is the maximum number of edges it can have? How about in dimension 3?.

23. If the graphs of 3-polytopes P, Q are the same can one conclude they are the same polytope? Does this hold in dimension 4?

24. Show that there are only 5 different Platonic Solids (a Platonic solid is a convex 3-dimensional polytope all of whose facets are the same regular polygon).

25. From (B) Chapter 7 exercises Page 66 problems 2,3,4,5,7,9,11.

Computational Problems (Computer is necessary is most cases now)

1. Let

$$P = conv(\{(-1, 3, 1, 2), (-1, 3, -1, 1), (-1, -1, 1, 1), (-1, -1, -1, 0),$$

$$(3, -1, 1, -3), (3, -1, -1, 4)\})$$

. Determine all the faces of P and of the polar P^o and draw the graphs of P and that of its polar. What are the facets? How many are there? Draw its Schlegel diagram.

2. Find all extreme points and facets of the polyhedron $M_{3\times 3}(1)$. Do a Schlegel diagram of this 4-dimensional polyhedron.

3. Compute a random 4-dimensional polytope P as the convex hull of 10 random points using `rand_sphere(4,10)`. Run `VISUAL` to see a Schlegel diagra m. How many 3-dimensional polytopes do you see? How many facets does P have?

4. Starting in dimension three, construct a polytope as the convex hull of a collection of points using the `polymake` function `POINTS`. How many vertices (use `N_VERTICES`) can you get? Try to get the largest number possible. Ho w many facets can you get?

5. The cyclic polytope $C(n, d)$ is constructed as follows: From the *moment curve* defined as

$$\varphi(t) = (t^d, t^{d-1}, \dots, t^2, t).$$

Pick any n distinct values for $t_1 < t_2 < \cdots < t_n$. Then a polytope $C(n, d)$ is

$$conv(\varphi(t_1), \varphi(t_2), ldots, \varphi(t_n)).$$

Prove that the cyclic polytope has the property that all its faces are simplices (i.e., it is a simplicial polytope). Experiment with drawing it for several choices of t_i. What do you notice?

6. Compute all the facets of a cyclic polytope $C(8, 4)$ What is its graph? What is its polar? How many facets?

7. Draw the Schlegel diagram of an icosahedron. Draw a Schlegel diagram of the Cartesian product of two triangles.

8. Consider the 24-cell described in exercise 8.5 of the Chakerian notes. Figure out everything you can about it: Dimension, number of facets, number of vertices, graph, schlegel diagram, volume, polar polytope, etc. Figure out what you get when you slice it through with the hyperplane $x_1 + x_2 - x_3 - x_4 = 0$. Draw a picture of the resulting polytope!

9. Draw a sequence of 4 slices of the 4-dimensional cube using hyperplanes perpendicular to the direction that connects the opposite vertices $(-1,-1,-1,-1)$ and $(1,1,1,1)$.

10. From (A) Chapter 3 Show using Farkas lemma that the system of equations and inequalities

$$\begin{aligned} x+2y+3z+w &= 2 \\ 3x+y+5z+w &= 1 \\ x+2y+z+w &= 1 \\ x &\geq 0 \\ y &\geq 0 \\ z &\geq 0 \\ w &\geq 0 \end{aligned}$$

has no real solutions (HINT: linear algebra will NOT work here, why?).

11. Consider the polyhedron P defined by the following system of inequalities:

$$\begin{aligned} -x-4y+4z &\leq 9 \\ -2x-y-3z &\leq -4 \\ x-2y+5z &\leq 0 \\ x-z &\leq 4 \\ 2x+y-2z &\leq 11 \\ -2x+6y-5z &\leq 17 \\ -6x-y+8z &\leq -6. \end{aligned}$$

Use Fourier-Motzkin elimination to eliminate the variable y. What is the "shadow" of the polyhedron under the projection? What are the smallest and largest values of x? Draw the polytope P to confirm this. How do (the coordinates of) the vertices confirm this same information?

12. Find all integer solutions x, y, z of to the system of inequalities

$$\begin{aligned} -x+y-z &\leq 0 \\ -y+z &\leq 0 \\ -z &\leq 0 \\ x-z &\leq 1 \\ y &\leq 1 \\ z &\leq 1 \end{aligned}$$

13. Let Q be the polyhedron (a polygon) given by the inequalities:

$$\begin{aligned} -x - y &\leq 0 \\ 2x - y &\leq 1 \\ -x + 2y &\leq 1 \\ x + 2y &\leq 2 \end{aligned}$$

Compute all vertices and edges of the polytope. Then check this corresponds to the intuitive notion for polygons.