Convex Geometry: MATH 114

The following list collects all the problems on which you will be examined for the second midterm. There will be again a take-home midterm portion and an in-class portion. Those problems after the line will be part of the final.

Exercises for second midterm & Final

- **Problems for Second Midterm**

  1. Prove Carathéodory's Theorem If $x \in conv(S)$ in $\mathbb{R}^d$, then x is the convex combination of $d+1$ points in S.
  2. Learn one proof of Radon and Helly theorems. Make sure you know how to explain it well.
  3. Every vertex of a polyhedron is an extreme point.
  4. Consider the polytope P defined by the following system of inequalities:

$$\begin{aligned} -x - 4y + 4z &\leq 9 \\ -2x - y - 3z &\leq -4 \\ x - 2y + 5z &\leq 0 \\ x - z &\leq 4 \\ 2x + y - 2z &\leq 11 \\ -2x + 6y - 5z &\leq 17 \\ -6x - y + 8z &\leq -6. \end{aligned}$$

     Use Fourier-Motzkin elimination to eliminate the variable y. What is the "shadow" of the polyhedron under the projection? What are the smallest and largest values of x? Draw the polytope P to confirm this. How do (the coordinates of) the vertices confirm this same information?
  5. Find all integer solutions x, y, z of to the system of inequalities

$$\begin{aligned} -x + y - z &\leq 0 \\ -y + z &\leq 0 \\ -z &\leq 0 \\ x - z &\leq 1 \\ y &\leq 1 \\ z &\leq 1 \end{aligned}$$

  6. Let Q be the polyhedron (a polygon) given by the inequalities:

$$
\begin{aligned}
-x - y &\leq 0 \\
2x - y &\leq 1 \\
-x + 2y &\leq 1 \\
x + 2y &\leq 2
\end{aligned}
$$

Using the theorems seen in class compute all vertices and edges of the polytope. Then check this corresponds to the intuitive notion for polygons.

7. Use what you know about convex sets to prove that a bounded polyhedron is a polytope (Hint: Krein-Milman and the description of faces for polyhedra!)

8. Let $P = \{x \in R^d : Ax \leq d\}$ be a polyhedron. Prove that if P contains a line $\{v + tu : t \in R\}$ with u non-zero directional vector, then $Au = 0$.

9. Prove that the recession cone of a polyhedron $P = \{x \in R^d : Ax \leq d\}$ is equal to $\{x \in R^d : Ax \leq 0\}$.

10. Show using Farkas lemma that the system of equations and inequalities

$$
\begin{aligned}
x + 2y + 3z + w &= 2 \\
3x + y + 5z + w &= 1 \\
x + 2y + z + w &= 1 \\
x &\geq 0 \\
y &\geq 0 \\
z &\geq 0 \\
w &\geq 0
\end{aligned}
$$

has no real solutions (HINT: linear algebra will NOT work here, why?).

11. Prove the following version of Farkas' lemma (HINT: use the version we proved in class as a lemma) $\{x : Ax \leq b, x \geq 0\} \neq \emptyset \iff$ When $y^T A \geq 0,$ then $y^T b \geq 0$

12. Give an example of non-convex cone

13. Prove in detail that the set

$$C = \{(x, y, z) : z \geq 0, x^2 + y^2 \leq z^2\}$$

is a convex cone. Is it finitely generated?

14. Let $P = conv(v_1, v_2, \ldots, v_m)$ and also has an inequality representation $P = \{x : Ax \leq b\}$ with A a $m \times d$ matrix. Prove that $conv(v_i, v_j)$ is a 1-dimensional face of P if the rank of the matrix $A_I(z)$ is $d-2$ where $z = 1/2(v_i + v_j)$.

15. Let C be a finitely generated cone. Prove that the polar of the polar of C equals C.

- **Problems for Final Exam**

1. For each of the following cases either give a 3-polytope having the proposed f-vector or tell why there is no such polytope. $(f_0, f_1, f_2) = (27, 55, 26)$, $(f_0, f_1, f_2) = (12, 23, 13)$,$(f_0, f_1, f_2) = (35, 51, 18)$.

2. Suppose that every vertex of a 3-polytope is 4-valent. Find an equation for v (number of vertices) in terms of e (number of edges).

3. Describe an infinite family of 3-polytopes all of whose facets are 4-sided polygons.

4. Prove that no 3-polytope has exactly 7 edges.

5. Prove that for any $n \geq 6$ and $n \neq 7$ there exists a 3-polytope with exactly n edges.

6. Let v_i the number of i-valent vertices of a 3-polytope. Estimate $\sum_i i * v_i$. What can be said of the quantity $\sum_i (6-i)v_i$? HINT: You should get inequalities involving faces and/or edge of the polytope.

7. A 3-polytope is simplicial iff each facet is a triangle. Show that the inequalities of the previous problem turn into equations.

8. Prove that for any 3-polytope $\sum_i (4-i)(v_i + p_i) = 8$ HINT: use your knowledge about $\sum_i 4v_i$ and $\sum_i 4p_i$.

9. Does there exist a 3-polytope whose vertices are all 4-valent and whose facets are all quadrilaterals?

10. Does there exist a 3-polytope for which each two facets have a different number of edges?

11. Show that if P is a 3-polytope such that each facet is a regular triangle, then P has at most 12 vertices and at most 20 facets. Then prove that there is no such 3-polytopes with 18 facets.

12. If a d-polytope has V vertices, what is the maximum number of edges it can have? How about in dimension 3?.

13. If the graphs of 3-polytopes P, Q are the same can one conclude they are the same polytope? Does this hold in dimension 4?

14. Show that there are only 5 different Platonic Solids (a Platonic solid is a convex 3-dimensional polytope all of whose facets are the same regular polygon).