Math and Computers, Math 165
Homework three, Due Nov 7

1. The basis of an ideal is different from a basis in linear algebra in that we do not care about linear independence! As a consequence when we write an element $f \in \langle f_1, \ldots, f_s \rangle$ as $f = \sum h_i f_i$ the coefficients h_i are not always unique. As an example, write $x^2 + xy + y^2 \in \langle x, y \rangle$ in two different ways.

2. Each of the following polynomials is written with its monomials ordered according to exactly one of the monomial orders: Lex, graded lex, or graded reverse lex. Determine which monomial order was used in each case.

   (a) $7x^2y^4z - 2xy^6 + x^2y^2$ (b) $xy^3z + xy^2z^2 + x^2z^3$ (c) $x^4y^5z + 2x^3y^2z - 4xy^2z^4$

3. show that graded reverse lexicographic order is indeed a monomial order.

4. Let $>$ be a monomial order in $S = \mathbb{C}[x_1, \ldots, x_n]$.

   (a) Let $f \in S$ and let m be a monomial. Show that $LT(m \cdot f) = m \cdot LT(f)$.

   (b) Let $f, g \in S$. Is $LT(f \cdot g)$ necessarily the same as $LT(f) \cdot LT(g)$?

5. Applying the multivariate division algorithm by hand, and using graded lexicographic order compute

   (a) the remainder of $f = xy^2z^2 + xy - yz$ of division by the polynomials $\{f_1 = x - y^2, f_2 = y - z^3, f_3 = z^2 - 1\}$ given in the order (f_1, f_2, f_3)

   (b) Repeat part (a) with the order of the dividing polynomials now (f_2, f_3, f_1).

   (c) the remainder of $f = x^3 - x^2y - x^2z$ by the polynomials $f_1 = x^2y - z$ and $f_2 = xy - 1$. First find the remainder of division by (f_1, f_2), then find the remainder of division by (f_2, f_1). What do you notice?

6. **Programming Assignment:** Write a computer program that plots the complex roots of a univariate polynomial f and those of its derivative f'. The roots must be labeled differently (by symbol) to distinguish from which polynomial they come from. Then compute the convex hull of (a) all the points (b) just roots of f, (c) just roots of f'. Make sure to draw the line segments defining the convex hull. Experiment with at least 25 different polynomials of various degrees and make careful observations about the convex hull and the points in the boundary.