Enumerative Combinatorics, Math 245
Homework four

(1) Problems from Stanley's book Chapter 3: 5, 10a, 11, 23a, 44, 45, 53a, 56ab
(2) Consider the lattice of subspaces of the vector space $(F_q)^n$, where F_q is a finite field. Prove that the size of the largest antichain is at most $\left[{n \atop \lfloor n/2 \rfloor} \right]$ (a q-binomial coefficient).
(3) Let $A(P)$ denote the incidence algebra of the poset P Let $\eta = \zeta - \delta$. Show that $\eta^k(a, b)$ counts the number of chains from a to b of length k. Determine η^k for the Boolean poset $B(n)$.
(4) Show that the Möbius function μ is equal to $\sum_{k \geq 0} (-1)^k \eta^k$ inside the incidence algebra $A(P)$. Verify this for the Boolean poset.
(5) Let $P(Q)$ be the face lattice of a convex polytope Q with the smallest element o deleted. Let $O(P(Q))$ be its order complex, if we think of the simplices of $O(P(Q))$ as simplices made of points of Q can you find a geometric representation of $O(P(Q))$ in Q.
(6) Given a poset P with elements $x_1, ..., x_n$ one can define an *order polytope* $Q(P)$ in R^n by: $Q(P) = \{X \in R^n : 0 \leq X_i <= 1, X_i > X_j \quad if \quad x_i > x_j \quad in \quad P\}$. Prove that the vertices of $Q(P)$ are in bijection to the order ideals of P. What are the linear extensions of P in terms of Q(P)?