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Vertices of Gelfand–Tsetlin Polytopes*

Jesús A. De Loera and Tyrrell B. McAllister

Department of Mathematics, University of California at Davis,
Davis, CA 95616, USA
{tmcal,deloera}@math.ucdavis.edu

Abstract. This paper is a study of the polyhedral geometry of Gelfand–Tsetlin polytopes arising in the representation theory of $\mathfrak{gl}_n\mathbb{C}$ and algebraic combinatorics. We present a combinatorial characterization of the vertices and a method to calculate the dimension of the lowest-dimensional face containing a given Gelfand–Tsetlin pattern.

As an application, we disprove a conjecture of Berenstein and Kirillov [1] about the integrality of all vertices of the Gelfand–Tsetlin polytopes. We can construct for each $n \geq 5$ a counterexample, with arbitrarily increasing denominators as n grows, of a nonintegral vertex. This is the first infinite family of nonintegral polyhedra for which the Ehrhart counting function is still a polynomial. We also derive a bound on the denominators for the nonintegral vertices when n is fixed.

1. Introduction

Many authors have recently observed that polyhedral geometry plays a special role in combinatorial representation theory (see, for example, [2], [7], [8], [10]–[12], [14], and the references within). In this note we study the polyhedral geometry of the so-called Gelfand–Tsetlin patterns, which arise in the representation theory of $\mathfrak{gl}_n\mathbb{C}$ and the study of Kostka numbers.

For each $n \in \mathbb{N}$, let X_n be the set of all triangular arrays $(x_{ij})_{1\leq i\leq j\leq n}$ with $x_{ij} \in \mathbb{R}$. Then X_n inherits a vector space structure under the obvious isomorphism $X_n \cong \mathbb{R}^{n(n+1)/2}$.

Definition 1.1. A *Gelfand–Tsetlin pattern* or *GT-pattern* is a triangular array $(x_{ij})_{1\leq i\leq j\leq n} \in X_n$ satisfying the following inequalities:

- $x_{ij} \geq 0$, for $1 \leq i \leq j \leq n$; and
- $x_{i,j+1} \geq x_{ij} \geq x_{i+1,j+1}$, for $1 \leq i \leq j \leq n-1$.

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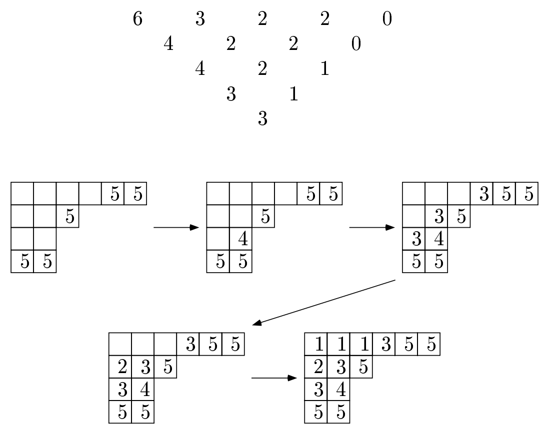

Fig. 1. A bijection mapping $GT(\lambda, \mu) \cap Z^{(n(n+1)/2)} \to SSYT(\lambda, \mu)$.

We always depict a GT-pattern $(x_{ij})_{1 \le i \le j \le n}$ by arranging the entries as follows:

$$\begin{array}{ccccccccc} x_{1n} & & \cdots & & \cdots & & \cdots & & x_{nn} \\ & \ddots & & \ddots & & \iddots & & \iddots & \\ & & x_{13} & & x_{23} & & x_{33} & & \\ & & & x_{12} & & x_{22} & & & \\ & & & & x_{11} & & & & \end{array}$$

In this arrangement, the inequalities in Definition 1.1 state that each entry is nonnegative, and each entry not in the top row is weakly less than its upper-left neighbor and weakly greater than its upper-right neighbor. We refer to the elements $x_{1j}, \ldots, x_{jj}$ as the jth row, i.e., the jth row *counted from the bottom*. The solutions of these inequalities define a polyhedral cone in $\mathbb{R}^{n(n+1)/2}$. See the top of Fig. 1 for an example of a GT-pattern.

Definition 1.2. Given $\lambda, \mu \in \mathbb{Z}^n$, the *Gelfand–Tsetlin polytope* $GT(\lambda, \mu) \subset X_n$ is the convex polytope of GT-patterns $(x_{ij})_{1 \le i \le j \le n}$ satisfying the equalities

- $x_{in} = \lambda_i$, for $1 \le i \le n$;
- $x_{11} = \mu_1$; and $\sum_{i=1}^{j} x_{ij} - \sum_{i=1}^{j-1} x_{i,j-1} = \mu_j$, for $2 \le j \le n$.

In other words, $GT(\lambda, \mu)$ is the set of all GT-patterns in X_n in which the top row is λ and the sum of the entries in the jth row is $\sum_{i=1}^{j} \mu_i$ for $1 \le j \le n$. Note that when we speak of a GT-polytope $GT(\lambda, \mu)$, we assume that λ and μ are integral.

The importance of GT-polytopes stems from a classic result of Gelfand and Tsetlin in [6], which states that the number of integral lattice points in the GT-polytope $GT(\lambda, \mu)$ equals the dimension of the weight μ subspace of the irreducible representation of $\mathfrak{gl}_n\mathbb{C}$ with highest weight λ. These subspaces have bases indexed by the set $SSYT(\lambda, \mu)$ of semistandard Young tableaux with shape λ and content μ [18]. It is well known that the elements of $SSYT(\lambda, \mu)$ are in one-to-one correspondence with the integral GT-patterns in $GT(\lambda, \mu)$ under the bijection exemplified in Fig. 1: Given an integral GT-pattern in

X_n, let $\lambda^{(j)}$ be the jth row (so that $\lambda^{(n)} = \lambda$). For $1 \le j \le n$, place a j in each of the boxes in the skew shape $\lambda^{(j)}/\lambda^{(j-1)}$ in the Young diagram of shape λ. (Here we put $\lambda^{(0)} = \emptyset$ to deal with the $j = 1$ case.) See [18] for details and [1] and [8] for more interesting uses of GT-polytopes. Now we introduce the main combinatorial tool for the study of vertices of the GT-polytopes:

Definition 1.3. Given a GT-pattern $\mathbf{x} \in X_n$, the *tiling* $\mathcal{P}$ of $\mathbf{x}$ is the partition of the set

$$\{(i, j) \in \mathbb{Z}^2 \colon 1 \le i \le j \le n\}$$

into subsets, called *tiles*, that results from grouping together those entries in $\mathbf{x}$ that are equal and adjacent. More precisely, $\mathcal{P}$ is that partition of $\{(i, j) \in \mathbb{Z}^2 \colon 1 \le i \le j \le n\}$ such that two pairs (i, j), $(\tilde{i}, \tilde{j})$ are in the same tile if and only if there are sequences

$$\begin{aligned} i &= i_1, i_2, \ldots, i_r = \tilde{i}, \\ j &= j_1, j_2, \ldots, j_r = \tilde{j} \end{aligned}$$

such that, for each $k \in \{1, \ldots, r - 1\}$, we have that

$$(i_{k+1}, j_{k+1}) \in \{(i_k + 1, j_k + 1), (i_k, j_k + 1), (i_k - 1, j_k - 1), (i_k, j_k - 1)\}$$

and $x_{i_{k+1} j_{k+1}} = x_{i_k j_k}$.

In other words, the tiles are just the connected components in the diagram of a GT-pattern, where two entries are connected when they are adjacent and contain the same value. See Fig. 2 for examples of GT-patterns and their tilings. The shading of some of the tiles in that figure is explained below.

Given a GT-pattern $\mathbf{x}$ with tiling $\mathcal{P}$, we associate to $\mathcal{P}$ (or, equivalently, to $\mathbf{x}$) a matrix $A_{\mathcal{P}}$ as follows. Define the *free tiles* $P_1, P_2, \ldots, P_s$ of $\mathcal{P}$ to be those tiles in $\mathcal{P}$ that do not intersect the bottom or top row of $\mathbf{x}$, i.e., those tiles that do not contain $(1, 1)$ and do not contain (i, n) for $1 \le i \le n$. The order in which the free tiles are indexed will not matter for our purposes, but, for concreteness, we adopt the convention of indexing the

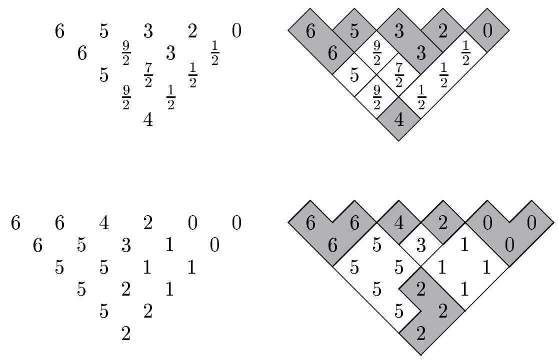

Fig. 2. Tilings of GT-patterns.

free tiles in the order that they are initially encountered as the entries of **x** are read from left to right and bottom to top. Define the *tiling matrix* $A_{\mathcal{P}} = (a_{jk})_{2\le j\le n-1,\,1\le k\le s}$ by

$$a_{jk} = \#\{i\colon (i,j) \in P_k\}.$$

(Note that the index j begins at 2.) That is, a_{jk} counts the number of entries in the jth row of **x** that are contained in the free tile P_k.

Example 1.4. Two GT-patterns and their tilings are given in Fig. 2. The unshaded tiles are the free tiles. The associated tiling matrices are respectively

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

The motivation for introducing tilings, and the main result of this paper, is the following.

Theorem 1.5. *Suppose that $\mathcal{P}$ is the tiling of a GT-pattern* **x**. *Then the dimension of the kernel of $A_{\mathcal{P}}$ is equal to the dimension of the minimal* (*dimensional*) *face of the GT-polytope containing* **x**.

As a corollary to this result, we get an easy-to-check criterion for a GT-pattern being a vertex of the GT-polytope containing it.

Corollary 1.6. *If* $\mathbf{x} \in GT(\lambda, \mu)$ *has tiling $\mathcal{P}$ containing s free tiles*, *then the following conditions are equivalent*:

- **x** *is a vertex of* $GT(\lambda, \mu)$; *and*
- $A_{\mathcal{P}}$ *has trivial kernel*; i.e, *for some $s \times s$ submatrix $\tilde{A}$ of $A_{\mathcal{P}}$*, $\det \tilde{A} \neq 0$.

As an application of Theorem 1.5, we present a solution to a conjecture by Berenstein and Kirillov (Conjecture 2.1 on p. 101 in [1]): all vertices of a GT-polytope have integer coordinates, i.e., $GT(\lambda, \mu)$ is a convex integral polytope. This conjecture seems to have been motivated by the fact that, for an integer parameter l, the Kostka number $K_{l\lambda,l\mu}$ is a univariate polynomial in l when λ and μ are fixed. This was proved by Kirillov and Reshetikhin using fermionic formulas in [9]. For completeness, we give another proof at the end of Section 2. Billey et al. [3] have shown that, more strongly, $K_{\lambda\mu}$ is a piecewise multivariate polynomial in λ and μ. It is natural to ask whether the above polynomial properties of the Kostka numbers extend to the *Littlewood–Richardson coefficients* $c^{\nu}_{\lambda,\mu}$. Indeed, Derksen and Weyman established that the one-parameter dilations of these numbers (i.e., $c^{l\nu}_{l\lambda,l\mu}$ with λ, μ, ν fixed) are again univariate polynomials [4]. Rassart (see [15]) has now extended the piecewise multivariate polynomiality of Kostka numbers to Littlewood–Richardson coefficients.

We must comment that it is quite natural to conjecture the integrality of the vertices of GT-polytopes, if one knows of the theory of Ehrhart functions that count the number of lattice points inside convex polytopes and their dilations (see Chapter 4 of [17]). The Ehrhart counting functions are known to be polynomials when the vertices are integral. As a consequence, in the following theorem we are in fact presenting the first infinite family of nonintegral polyhedra whose Ehrhart counting functions are still polynomials. Other low-dimensional families have been found recently [13]. Finally, we must remark that R.P. Stanley communicated to us that his student Peter Clifford noticed nonintegrality for GT-polytopes earlier (unpublished) and that King et al. had independently noticed nonintegrality for hive polytopes (which are isomorphic to GT-polytopes under a lattice-preserving linear map) in the case $n = 5$ (see [7]). They also proved integrality of vertices for $n \leq 4$, did a nice study of "stretched" Kostka and Littlewood–Richardson coefficients, and presented several conjectures again concerning the polynomiality of Ehrhart counting functions.

Theorem 1.7. *The Berenstein–Kirillov conjecture is true for $n \leq 4$, but counterexamples to this conjecture exist for all values of $n \geq 5$. More strongly, by choosing n sufficiently large, we can find GT-polytopes in which the denominators of the vertices are arbitrarily large: For positive integer k, let $\lambda = (k^k, k-1, 0^k)$ and $\mu = ((k-1)^{k+1}, 1^k)$. Then a vertex of $GT(\lambda, \mu) \subset X_{2k+1}$ contains entries with denominator k.*

2. Proof of the Main Result and Its Consequences

Proof. [Proof of Theorem 1.5] Suppose that $\mathcal{P}$ is the tiling of a GT-pattern $\mathbf{x}$ in the GT-polytope $GT(\lambda, \mu) \subset X_n$. Let s be the number of free tiles in $\mathcal{P}$. Let $(\varepsilon^{(1)}, \ldots, \varepsilon^{(d)})$ be a basis for $\ker A_{\mathcal{P}}$. Because we can scale the basis by any nonzero scalar, we can assume that

$$|\varepsilon^{(m)}_k| < 1/2 \min\{|x_{i_1 j_1} - x_{i_2 j_2}| : x_{i_1 j_1} \neq x_{i_2 j_2}\}, \qquad \text{for } 1 \leq m \leq d, \quad 1 \leq k \leq s,$$

where $\varepsilon^{(m)}_k$ is the kth coordinate of $\varepsilon^{(m)}$.

Let $H \subset X_n$ be the linear subspace of X_n such that $H + \mathbf{x}$ is the affine span of the minimal face of $GT(\lambda, \mu)$ containing $\mathbf{x}$. Define a linear map $\varphi: \ker A_{\mathcal{P}} \to X_n$ by $\varphi(\varepsilon^{(m)}) = \mathbf{y}^{(m)}$, where

$$y^{(m)}_{ij} = \begin{cases} \varepsilon^{(m)}_k & \text{if } (i, j) \text{ is in the free tile } P_k \text{ of } \mathcal{P}, \\ 0 & \text{if } (i, j) \text{ is not in a free tile of } \mathcal{P}. \end{cases}$$

(See Example 2.1.) Thus, $\mathbf{x} + \mathbf{y}^{(m)}$ is the result of adding $\varepsilon^{(m)}_k$ to each entry in the kth free tile of $\mathbf{x}$ for $1 \leq k \leq s$.

The claim is that $(\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(d)})$ is a basis for H. First, since the $\varepsilon^{(m)}_k$'s are sufficiently small, $\mathbf{x} \pm \mathbf{y}^{(m)}$ is a GT-pattern. Moreover, $y^{(m)}_{11} = 0$, $y^{(m)}_{in} = 0$ for $1 \leq i \leq n$, and each row-sum of $\mathbf{y}^{(m)}$ is 0. This last fact is true because $\varepsilon^{(m)} \in \ker A_{\mathcal{P}}$ and the row-sum is, by construction, the same as the dot product of $\varepsilon^{(m)}$ with the matrix $A_{\mathcal{P}}$. Taken together, these properties yield that $\mathbf{x} \pm \mathbf{y}^{(m)} \in GT(\lambda, \mu)$. That is, $\mathbf{x} + \mathbf{y}^{(m)}$ and $\mathbf{x} - \mathbf{y}^{(m)}$ are the endpoints of a line segment contained in $GT(\lambda, \mu)$ that contains $\mathbf{x}$ in its relative interior. This establishes that $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(d)} \in H$.

That $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(d)}$ are linearly independent clearly follows from the fact that $\varepsilon^{(1)}, \ldots, \varepsilon^{(d)}$ are linearly independent. Thus, it remains only to prove that $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(d)}$ span H. Suppose that $\mathbf{y} \in H$, and assume that $\mathbf{y}$ is scaled by a nonzero amount so that $\mathbf{x} \pm \mathbf{y} \in GT(\lambda, \mu)$. We construct an element ε of $\ker A_{\mathcal{P}}$ such that $\varphi(\varepsilon) = \mathbf{y}$. Note that

- $y_{ij} = 0$ when (i, j) is in the bottom or top row of $\mathcal{P}$,
- each row-sum of $\mathbf{y}$ is 0, and
- if (i_1, j_1) and (i_2, j_2) are in the same tile of $\mathcal{P}$, then $y_{i_1 j_1} = y_{i_2 j_2}$.

To see that this last property holds, it suffices (see Definition 1.3) to examine the case where $y_{i_1 j_1}$ and $y_{i_2 j_2}$ are adjacent entries, i.e., where

$$(i_2, j_2) \in \{(i_1 + 1, j_1 + 1), (i_1, j_1 + 1), (i_1 - 1, j_1 - 1), (i_1, j_1 - 1)\}.$$

Since $\mathbf{x} \pm \mathbf{y}$ is a GT-pattern (see Definition 1.1), we must have either

$$x_{i_1 j_1} + y_{i_1 j_1} \leq x_{i_2 j_2} + y_{i_2 j_2} \qquad \text{and} \qquad x_{i_1 j_1} - y_{i_1 j_1} \leq x_{i_2 j_2} - y_{i_2 j_2}$$

or

$$x_{i_1 j_1} + y_{i_1 j_1} \geq x_{i_2 j_2} + y_{i_2 j_2} \qquad \text{and} \qquad x_{i_1 j_1} - y_{i_1 j_1} \geq x_{i_2 j_2} - y_{i_2 j_2}.$$

However, since (i_1, j_1) and (i_2, j_2) are in the same tile of $\mathcal{P}$, we have $x_{i_1 j_1} = x_{i_2 j_2}$. Thus, in either case, we can subtract the $\mathbf{x}$ entries from both sides, yielding $y_{i_1 j_1} = y_{i_2 j_2}$, as claimed.

For $1 \leq k \leq s$ and for each (i, j) in the free tile P_k, put $\varepsilon_k = y_{ij}$. Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_s)$. Then, from the conditions on $\mathbf{y}$ given above, $\varepsilon \in \ker A_{\mathcal{P}}$ and $\varphi(\varepsilon) = \mathbf{y}$. Hence, the coordinates of ε with respect to the basis $(\varepsilon^{(1)}, \ldots, \varepsilon^{(d)})$ of $\ker A_{\mathcal{P}}$ will also be the coordinates of $\mathbf{y}$ with respect to $(\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(d)})$. In particular, $(\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(d)})$ is a basis for H, as claimed. □

Example 2.1. Let $\mathbf{x}$ be the GT-pattern

$$\begin{array}{ccccccccc}
6 & & 5 & & 3 & & 2 & & 0 \\
& 6 & & \frac{9}{2} & & 3 & & \frac{1}{2} & \\
& & 5 & & \frac{7}{2} & & \frac{1}{2} & & \\
& & & \frac{9}{2} & & \frac{1}{2} & & & \\
& & & & 4 & & & &
\end{array}$$

from Fig. 2. We explicitly apply to $\mathbf{x}$ the constructions in the proof of Theorem 1.5. This GT-pattern has tiling matrix

$$A_{\mathcal{P}} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

A "sufficiently short" basis for $\ker A_{\mathcal{P}}$ is

$$(\varepsilon^{(1)}, \varepsilon^{(2)}) = \left(\tfrac{1}{3}\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \tfrac{1}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right).$$

(Here, "sufficiently short" refers to the fact that $\mathbf{x} + \mathbf{y}^{(1)}$ and $\mathbf{x} + \mathbf{y}^{(2)}$, which are constructed shortly, will lie within the minimal face containing $\mathbf{x}$.) Therefore, $\mathbf{x}$ lies in a two dimensional face of

$$GT((6, 5, 3, 2, 0), (4, 1, 4, 5, 2)).$$

Applying the map φ from the proof to $(\varepsilon^{(1)}, \varepsilon^{(2)})$ yields

$$\mathbf{y}^{(1)} = \begin{array}{ccccccccc} 0 & & 0 & & 0 & & 0 & & 0 \\ & 0 & & 0 & & 0 & & 0 & \\ & & -\frac{1}{3} & & \frac{1}{3} & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 0 & & & & \end{array}$$

and

$$\mathbf{y}^{(2)} = \begin{array}{ccccccccc} 0 & & 0 & & 0 & & 0 & & 0 \\ & 0 & & \frac{1}{3} & & 0 & & -\frac{1}{3} & \\ & & \frac{1}{3} & & 0 & & -\frac{1}{3} & & \\ & & & \frac{1}{3} & & -\frac{1}{3} & & & \\ & & & & 0 & & & & \end{array}$$

From the proof just given, the set $\{\mathbf{x}, \mathbf{x} + \mathbf{y}^{(1)}, \mathbf{x} + \mathbf{y}^{(2)}\}$ affinely spans the affine hull of the minimal face containing x.

The machinery of tilings allows us easily to find nonintegral vertices of GT-polytopes by looking for a tiling with a tiling matrix satisfying certain properties given below. Then the tiling can be "filled" in a systematic way with the entries of a GT-pattern that is a nonintegral vertex.

Lemma 2.2. *Suppose that $\mathcal{P}$ is a tiling with s free tiles such that $A_{\mathcal{P}}$ has a trivial kernel. Then the following conditions are equivalent*:

(1) *$\mathcal{P}$ is the tiling of a nonintegral vertex $\mathbf{x}$ of a GT-polytope in which $q \in \mathbb{N}$ is the least common multiple of the denominators of the entries in $\mathbf{x}$ (written in reduced form); and*

(2) *there is an integral vector $\xi = (\xi_1, \ldots, \xi_s)$ such that $A_{\mathcal{P}}\xi \equiv 0 (\operatorname{mod} q)$ and such that, for some $k \in \{1, \ldots, s\}$, $\gcd(\xi_k, q) = 1$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $\mathbf{x}$ is a nonintegral vertex in which q is the least common multiple of the denominators of the entries. For each entry x_{ij}, $1 \leq i \leq j \leq n$, let

$p_{ij} = qx_{ij}$. Let $P_1, \ldots, P_s$ be the free tiles of $\mathcal{P}$, and define $\xi = (\xi_1, \ldots, \xi_s)$ by $\xi_k = p_{ij}$ for some $(i, j) \in P_k$ (all values of p_{ij} are equal within a tile). Since $\mathbf{x}$ has entries with denominator q, we have that, for some $k \in \{1, \ldots, s\}$, $\gcd(\xi_k, q) = 1$. Moreover, since each row-sum of $\mathbf{x}$ is an integer, we have that, for each *fixed* $j \in \{1, \ldots, n\}$,

$$q \text{ divides } \sum_{\substack{1 \le k \le s \\ (i,j) \in P_k}} p_{ij} = \sum_{1 \le k \le s} a_{jk}\xi_k.$$

Therefore, $A_{\mathcal{P}}\xi \equiv 0 (\bmod q)$.

(2) ⇒ (1) $\mathcal{P}$ is given to be a tiling, so some GT-pattern $\tilde{\tilde{\mathbf{x}}}$ with rational entries has tiling $\mathcal{P}$. If necessary, multiply $\tilde{\tilde{\mathbf{x}}}$ by some integer to produce an integral GT-pattern $\tilde{\mathbf{x}}$ with tiling $\mathcal{P}$. Choose $\xi = (\xi_1, \ldots, \xi_s)$ satisfying condition (2) such that $0 \le \xi_1, \ldots, \xi_s < q$. Define $\mathbf{y} \in X_n$ by

$$y_{ij} = \begin{cases} \xi_k / q & \text{if } (i, j) \text{ is in the free cell } P_k \text{ of } \mathcal{P}, \\ 0 & \text{if } (i, j) \text{ is not in a free cell of } \mathcal{P}. \end{cases}$$

Then $\mathbf{x} = \tilde{\mathbf{x}} + \mathbf{y}$ satisfies condition (1). □

Now we are ready to give the details of the proof of Theorem 1.7. In particular, Propositions 2.3 and 2.4 settle the Berenstein–Kirillov conjecture. Proposition 2.3 has also been proven by King et al. [7] with respect to hive polytopes, which are isomorphic to GT-polytopes under a lattice-preserving linear map. We give here a "tiling" proof.

Proposition 2.3. *When $n \le 4$, every GT-polytope in X_n is integral.*

Proof. Note that it suffices to prove the $n = 4$ case since there is a natural embedding $X_n \hookrightarrow X_{n+1}$ defined by $\mathbf{x} \mapsto \tilde{\mathbf{x}}$, where

$$\tilde{x}_{ij} = \begin{cases} 0 & \text{if } 1 \le i = j \le n+1, \\ x_{i,j-1} & \text{if } 1 \le i < j \le n+1. \end{cases}$$

Suppose that $\mathbf{x} \in X_4$ is a vertex. Then, by Corollary 1.6, the associated tiling matrix $A_{\mathcal{P}}$ has a trivial kernel. Therefore, $A_{\mathcal{P}}$ is either a 2×1 or a 2×2 matrix. Note also that the first and last nonzero entries of each column of a tiling matrix associated with a GT-pattern must be 1. Therefore, $A_{\mathcal{P}}$ is a 0/1-matrix.

If $A_{\mathcal{P}}$ is 2×1, then the only possibilities are

$$A_{\mathcal{P}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad A_{\mathcal{P}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{or} \quad A_{\mathcal{P}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

In each case, there exists no vector $\xi \not\equiv 0 (\bmod q)$ such that $A_{\mathcal{P}}\xi \equiv 0 (\bmod q)$ for $q > 1$, so Lemma 2.2 implies that the entries of $\mathbf{x}$ are integral. On the other hand, if $A_{\mathcal{P}}$ is 2×2, then $\det A_{\mathcal{P}} \in \{-1, 1\}$, i.e., $\gcd(\det A_{\mathcal{P}}, q) = 1$ for $q > 1$. Therefore, $A_{\mathcal{P}}$, considered as a module homomorphism on $\mathbb{Z}/q\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$, is invertible for $q > 1$, so, by Lemma 2.2, $\mathbf{x}$ is integral. □

Now we show that nonintegral GT-polytopes exist in X_n for each $n \geq 5$. Moreover, by choosing n sufficiently large, we can find GT-polytopes in which the denominators of the vertices are arbitrarily large.

Proposition 2.4. *For a positive integer k, let $\lambda = (k^k, k-1, 0^k)$ and $\mu = ((k-1)^{k+1}, 1^k)$. Then a vertex of $GT(\lambda, \mu) \subset X_{2k+1}$ contains entries with denominator k.*

Proof. Define $\mathbf{x}^{(k)} \in X_{2k+1}$ by

$$x_{ij}^{(k)} = \begin{cases} \frac{(k-j+1)(k+1)}{k} & \text{if} \quad 1 \leq i = j \leq k+1, \\ k - \frac{1}{k} & \text{if} \quad 1 \leq i < j \leq k+1, \\ k & \text{if} \quad k+1 < j \leq 2k+1 \quad \text{and} \quad 1 \leq i < j-k, \\ k - \frac{1}{k} & \text{if} \quad k+1 < j \leq 2k+1 \quad \text{and} \quad j-k \leq i \leq k, \\ \frac{(j-k-1)(k-1)}{k} & \text{if} \quad k+1 < j \leq 2k+1 \quad \text{and} \quad i = k+1, \\ 0 & \text{if} \quad k+1 < j \leq 2k+1 \quad \text{and} \quad k+1 < i \leq 2k+1. \end{cases}$$

(See Fig. 3.) Then $\mathbf{x}^{(k)} \in GT(\lambda, \mu)$. The tiling matrix associated with $\mathbf{x}^{(k)}$ is

$$A_{\mathcal{P}} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 2 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k-1 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ k & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ k-1 & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Since $\det A_{\mathcal{P}} = k$, $\mathbf{x}^{(k)}$ is a vertex of $GT(\lambda, \mu)$ by Corollary 1.6. □

Proposition 2.4 explicitly constructs counterexamples to the Berenstein–Kirillov conjecture in X_n where $n \geq 5$ is odd. Counterexamples with even $n \geq 6$ may be constructed

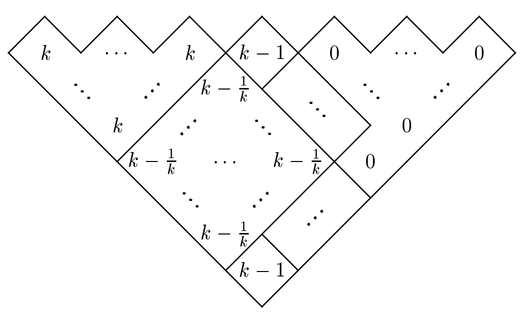

Fig. 3. An infinite family of counterexamples to the Berenstein–Kirillov conjecture.

from these using the embedding $X_n \hookrightarrow X_{n+1}$ given in the proof of Proposition 2.3. Less trivial examples with even n may be constructed using other tilings.

As a final application of tilings, we derive a bound on the size of the denominators in the vertices of GT-polytopes in fixed dimension. Observe that Lemma 2.2 says that if $\mathbf{x}$ is a nonintegral vertex in which q appears as a denominator, then the tiling matrix $A_{\mathcal{P}}$ has a trivial kernel as a linear operator $\mathbb{R}^s \to \mathbb{R}^{n-2}$ (since $\mathbf{x}$ is a vertex), but $A_{\mathcal{P}}$ has a *nontrivial* kernel when considered as an operator $(\mathbb{Z}/q\mathbb{Z})^s \to (\mathbb{Z}/q\mathbb{Z})^{n-2}$. Moreover, this nontrivial kernel contains a vector in which one of the coordinates is a unit in $\mathbb{Z}/q\mathbb{Z}$. This last condition implies that each $s \times s$ submatrix of $A_{\mathcal{P}}$ has determinant equal to 0 modulo q.

Proposition 2.5. *For fixed n, the numbers that may appear as denominators of entries in vertices of GT-polytopes in X_n are smaller than $(n-2)(n-1)!/4$.*

Proof. Fix $n \in \mathbb{N}$. Since only finitely many partitions of $\{(i, j) \in \mathbb{Z}^2 : 1 \le i \le j \le n\}$ exist, there is an upper bound on the set

$$\left\{ |m| : \begin{array}{l} m \text{ is the determinant of a square row submatrix} \\ \text{of the tiling matrix of some GT-pattern } \mathbf{x} \in X_n \end{array} \right\}.$$

By a "row submatrix", we mean a submatrix where the rows are a subset of the rows of the tiling matrix.

Let N be an upper bound on this set. The claim is that no GT-polytope in X_n has a vertex with denominators greater than N. Let $q > N$ be given. Suppose that $\mathbf{x} \in X_n$ is a vertex. Let s be the number of free tiles in $\mathbf{x}$, and let $A_{\mathcal{P}}$ be the tiling matrix of $\mathbf{x}$. Then no $s \times s$ submatrix of $A_{\mathcal{P}}$ has a determinant greater than or equal to q. Moreover, by Corollary 1.6, some $s \times s$ submatrix of $A_{\mathcal{P}}$ has a nonzero determinant. Therefore, this $s \times s$ submatrix has a determinant not equal to 0 modulo q. However, in the remarks preceding this proposition, we noted that if $\mathbf{x}$ is a vertex in which q is a denominator of one of the entries, then *every* $s \times s$ submatrix has a determinant equal to 0 modulo q. This proves that N is a bound as claimed.

Our second claim is that N is no more than $(n-2)(n-1)!/4$. All tiling matrices for GT-patterns in X_n have $n-2$ rows and only nonnegative entries. Moreover, since the first and last entry in each column must be a 1, and since each entry can differ by at most ± 1 from the entry above it, the largest possible entry in a tiling matrix is $(n-1)/2$. Therefore, if $A = (a_{ij})$ is an $s \times s$ submatrix of a tiling matrix, we have that

$$\det A \le \sum_{\sigma \in \mathfrak{A}_s} a_{1\sigma(1)} \cdots a_{s\sigma(s)} \le \frac{n-2}{4}(n-1)!,$$

where $\mathfrak{A}_s$ denotes the alternating group in $\mathfrak{S}_s$. □

The bound in Proposition 2.5 is not tight. For example, it is easy to show that, when $n = 5$, the largest possible denominator is $2 < (5-2)(5-1)!/4 = 18$.

To conclude this paper we present another proof of the following result:

Proposition 2.6. *Given a GT-polytope $GT(\lambda, \mu) \subset X_n$, the Ehrhart counting function $f(m) = \#(GT(m\lambda, m\mu) \cap \mathbb{Z}^{\binom{n+1}{2}})$ is a univariate polynomial.*

Proof. It is well known, from Ehrhart's fundamental work, that $f(m)$ must be a *quasipolynomial*. This means that there exist an integer M and polynomials $g_0, g_1, \dots, g_{M-1}$ such that $f(m) = g_i(m)$ if $m \equiv i (\mathrm{mod}\, M)$ (see details in Chapter 4 of [17]). So it is then enough to prove that, for some large enough value of m, a single polynomial interpolates all values from then on, because then the g_i's are forced to coincide infinitely many times, which proves that they are the same polynomial.

We use the algebraic meaning of $f(m)$ as the multiplicity of the weight $m\mu$ in the irreducible representation $V_{m\lambda}$ of $\mathfrak{gl}_n\mathbb{C}$. The well-known Kostant's multiplicity formula (see p. 421 of [5]) gives that

$$f(m) = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\varepsilon(\sigma)} K(\sigma(m\lambda + \delta) - m\mu - \delta), \qquad (*)$$

where $K(b)$ is Kostant's partition function for the root system A_n, $\varepsilon(\sigma)$ denotes the number of inversions of σ, and δ is one-half of the sum of the positive roots in A_n.

Kostant's partition function is what combinatorialists call a *vector partition function* [19]. More precisely, $K(b)$ is equal to the number of nonnegative integral solutions x of a linear system $Ax = b$. The columns of A are exactly the positive roots of the system A_n. Because the matrix A is unimodular [16], the counting function $K(b)$ is a multivariate piecewise polynomial function. The regions where $K(b)$ is a polynomial are convex polyhedral cones called *chambers* [19]. The chamber that contains b determines the polynomial value of $K(b)$; in fact it is the vector direction of b, not its norm, that determines the polynomial formula to be used.

In formula $(*)$ the right-hand side vector for Kostant's partition function is $b = \sigma(m\lambda + \delta) - (m\mu + \delta)$. As m grows, we might be moving from one chamber to another. Our claim is that, from some value of m on, the vectors $\sigma(m\lambda + \delta) - (m\mu + \delta)$ are inside the same chamber. To see this, note that in the expression $(*)$, μ, λ, and δ are constant vectors. For a given permutation σ, the vector direction $\sigma(m\lambda + \delta)$ is closer and closer to that of $\sigma(\lambda)$ when m grows in value. Similarly, the vector direction of $m\mu + \delta$ approaches that of μ when m grows. Thus, the direction of $b = \sigma(m\lambda + \delta) - (m\mu + \delta)$ approaches the direction of $b' = \sigma(\lambda) + \mu$ along a straight line. For sufficiently large m, the vectors b and b' are contained in the same chamber, where a single polynomial gives the value of $K(b)$.

We have shown that, for all values of m greater than some M, the formula $(*)$ represents an alternating sum of polynomials in the variable m. Therefore $f(m)$ is a polynomial, exactly as we wished to prove. □

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