

## The N-component real scalar field.

Let's describe a basic example of a gauge theory.  
For now, we'll take spacetime to be

$$M = \mathbb{R}^{3,1} = \text{Minkowski space with metric } \eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$x^\mu$  are the coordinates,  $\mu=0,1,2,3$ , with  $x^0 = t$ .

An N-component real scalar field is a map

$$\varphi: M \rightarrow \mathbb{R}^N.$$

A fundamental quantity for a field theory is the Lagrangian; a typical Lagrangian for  $\varphi$  looks like

$$\mathcal{L} = \eta^{\mu\nu} (\partial_\mu \varphi) \cdot (\partial_\nu \varphi) - V(\varphi \cdot \varphi)$$

inverse metric (has same components as metric, in this case)

$\partial_\mu := \frac{\partial}{\partial x^\mu}$

dot product in  $\mathbb{R}^N$

$V: \mathbb{R} \rightarrow \mathbb{R}$   
the "potential"

$\mathcal{L}$  will be a smooth function  $M \rightarrow \mathbb{R}$  if  $\varphi$  and  $V$  are smooth. But in physics we are interested in how  $\mathcal{L}$  changes when we change  $\varphi$ , so we think of the Lagrangian as

$$\mathcal{L}: \left[ \begin{array}{l} \text{all fields} \\ \varphi: M \rightarrow \mathbb{R}^N \end{array} \right] \rightarrow C^\infty(M).$$

$\mathbb{R}^N$  has an action of the rotation group

$$SO(N) = \left\{ \begin{array}{l} \text{orthogonal } N \times N \text{ matrices } U \\ \text{with } \det U = 1 \end{array} \right\}$$

and this action preserves the dot product:

$$Uv \cdot Uw = v \cdot w \quad U \in SO(N) \quad v, w \in \mathbb{R}^N$$

$SO(N)$  acts on the set of fields  $\varphi: M \rightarrow \mathbb{R}^N$  in the obvious way:

$$(U\varphi)(x) = U(\varphi(x))$$

$\uparrow$  matrix multiplication.

and  $\mathcal{L}$  is invariant under this action:

$$\begin{aligned} \mathcal{L}[U\varphi] &= \eta^{\mu\nu} (\partial_\mu U\varphi) \cdot (\partial_\nu U\varphi) - V(U\varphi \cdot U\varphi) \\ &= \eta^{\mu\nu} (U \partial_\mu \varphi) \cdot (U \partial_\nu \varphi) - V(U\varphi \cdot U\varphi) \\ &= \eta^{\mu\nu} (\partial_\mu \varphi) \cdot (\partial_\nu \varphi) - V(\varphi \cdot \varphi) = \mathcal{L}[\varphi]. \end{aligned}$$

The key idea of gauge theory is to let the rotation  $U$  vary from place to place in spacetime. I.e., instead of fixed  $U \in SO(N)$ , we pick a map

$$U: M \rightarrow SO(N)$$

and let this act on  $\varphi: M \rightarrow \mathbb{R}^N$  by the "gauge transformation"

$$\varphi(x) \mapsto U(x)\varphi(x).$$

We still get  $U\varphi \cdot U\varphi = \varphi \cdot \varphi$ , so the "potential" term in  $\mathcal{L}$  is invariant, but now

$$\partial_\mu(U\varphi) = U\partial_\mu\varphi + (\partial_\mu U)\varphi$$

so the "kinetic" term  $(\partial\varphi)^2$  won't be!

Trick: force the kinetic term to be invariant by changing  $\partial_\mu$  to  $D_\mu$  defined by

$$D_\mu\varphi = \partial_\mu\varphi + \underbrace{A_\mu}_{\text{some new } N \times N \text{ matrix-valued function on } M} \varphi$$

some new  $N \times N$   
matrix-valued function  
on  $M$

For this to have any chance of working, our gauge transformation

$$\varphi \mapsto U\varphi$$

will have to also act on  $A_\mu$  somehow. In fact, if we let  $U$  act by

$$A_\mu \mapsto UA_\mu U^{-1} + U\partial_\mu U^{-1}$$

then:

$$\begin{aligned} D_\mu\varphi &\mapsto \partial_\mu(U\varphi) + UA_\mu\varphi + U(\partial_\mu U^{-1})U\varphi \\ &= UD_\mu\varphi + (\partial_\mu U)\varphi + U(\partial_\mu U^{-1})U\varphi \end{aligned}$$

But  $0 = \partial_\mu(U^{-1}U) = (\partial_\mu U^{-1})U + U^{-1}(\partial_\mu U)$ , so two terms cancel...

... and we get

$$D_\mu \varphi \longmapsto U D_\mu \varphi$$

So, the modified Lagrangian

$$\mathcal{L} = \eta^{\mu\nu} (D_\mu \varphi) \cdot (D_\nu \varphi) - V(\varphi \cdot \varphi)$$

is invariant under gauge transformations, or "gauge invariant".

We have enlarged our naive group of symmetries of the theory,  $SO(N)$ , to the infinite dimensional group

$$C^\infty(M, SO(N)) = \text{gp. of smooth gauge transformations } M \rightarrow SO(N)$$

We did this at the cost of adding a new "field", the "gauge field"  $A_\mu$ . This gauge field takes on a life of its own in Yang-Mills theory, but that's a story for another time...

Easy exercise: Show that we may consistently take the matrices  $A_\mu$  to be antisymmetric. (We'll see this is no accident — it's because  $\mathfrak{so}(N)$ , the Lie algebra of  $SO(N)$ , consists of antisymmetric matrices.)

# Bundles

In our example, we "gauge" the  $SO(N)$  symmetry of our scalar field theory, allowing the symmetry group to act differently at each point. The mathematics behind this kind of gauge theory, and especially its generalizations, is the theory of "bundles". Let's explain the general ideas of bundle theory, and then see what they have to do with our  $SO(N)$  gauge theory.

If  $M$  is a smooth manifold, a bundle over  $M$  is just a smooth map

$$\begin{array}{ccc} E & \sim & \text{the } \underline{\text{total space}} \text{ of the bundle} \\ p \downarrow & & \\ M & \sim & \text{the } \underline{\text{base space}} \end{array}$$

We often say " $E$  is a bundle over  $M$ " for short, even though the bundle consists of  $E$  and the map  $p$ .

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The fiber over  $x \in M$  is the set  $E_x = p^{-1}\{x\}$ . Typically each fiber has the structure of some gadget (e.g. vector space, group, etc.) in which case, we say  $E$  is a "gadget bundle" or "bundle of gadgets" (e.g. a vector bundle, a bundle of groups, etc.)

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Given two bundles  $\begin{array}{c} E_1 \\ p_1 \downarrow \\ M_1 \end{array}$  and  $\begin{array}{c} E_2 \\ p_2 \downarrow \\ M_2 \end{array}$ , a bundle map is a

pair of smooth maps  $\alpha, \tilde{\alpha}$  s.t.

$$\begin{array}{ccc} E_1 & \xrightarrow{\tilde{\alpha}} & E_2 \\ p_1 \downarrow & & \downarrow p_2 \\ M_1 & \xrightarrow{\alpha} & M_2 \end{array} \quad \text{commutes, i.e. } p_2 \tilde{\alpha} = \alpha p_1.$$

This means precisely that  $\tilde{\alpha}$  maps the fiber over  $x \in M_1$  into the fiber over  $\alpha(x) \in M_2$ . When  $E_1$  and  $E_2$  are "gadget bundles", a "gadget bundle map" is a bundle map s.t. for each  $x \in M_1$ ,

$$\tilde{\alpha} : (E_1)_x \longrightarrow (E_2)_{\alpha(x)}$$

is a "gadgetomorphism", e.g.:

- a map of vector bundles maps fibers to fibers linearly
- a map of bundles of groups maps fibers to fibers via group homomorphisms

etc.

Next, a section of a bundle is a way of picking out an element of each fiber; if  $\begin{array}{c} E \\ p \downarrow \\ M \end{array}$  is a bundle, a (smooth) section is a (smooth) map  $\varphi : M \rightarrow E$ , drawn as:

$$\begin{array}{c} E \\ \downarrow \uparrow \varphi \\ M \end{array} \quad \text{such that } p\varphi = 1_M.$$

Now back to our  $N$ -component scalar field theory.  
 Before, we thought of  $\varphi$  as a function

$$\varphi: M \longrightarrow \mathbb{R}^N$$

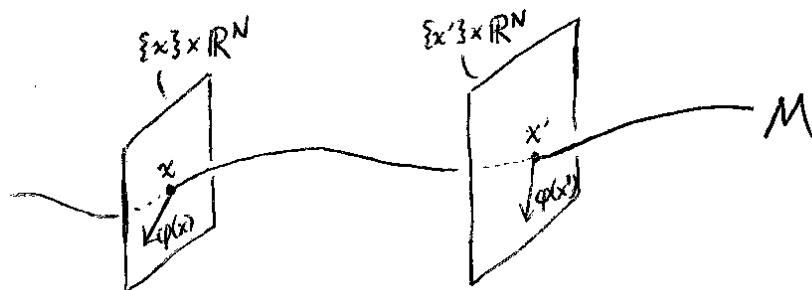
Now, we instead consider the vector bundle

$$\begin{array}{ccc} M \times \mathbb{R}^N & & (x, v) \\ \downarrow P & & \downarrow \\ M & & x \end{array}$$

called the trivial bundle on  $M$  with standard fiber  $\mathbb{R}^N$ ,  
 and think of  $\varphi$  as a section

$$\begin{array}{ccc} M \times \mathbb{R}^N & & \\ \downarrow P \quad \uparrow \varphi & & \\ M & & \end{array}$$

Here  $\varphi(x) = (x, \varphi_{\text{old}}(x))$  where  $\varphi_{\text{old}}: M \rightarrow \mathbb{R}^N$  is our  
 old version of  $\varphi$ . The idea is that  $\varphi(x)$  is not an  
 "element of  $\mathbb{R}^N$ " but an element of "the copy of  $\mathbb{R}^N$   
 at  $x$ ":



Each of these copies of  $\mathbb{R}^N$  can be rotated independently...

Before, we thought of a gauge transformation  $U$  as a function

$$U: M \longrightarrow SO(N)$$

Now, we think of it as a section of a bundle of groups:

$$\begin{array}{c} M \times SO(N) \\ \downarrow \quad \uparrow U \\ M \end{array}$$

where  $U(x) = (x, U_{\text{old}}(x))$ , of course.

A gauge transformation  $U: M \rightarrow M \times SO(N)$  acts on an  $N$ -component scalar field  $\varphi: M \rightarrow M \times \mathbb{R}^N$  in an obvious way:

$$\begin{aligned} (U\varphi): M &\longrightarrow M \times \mathbb{R}^N \\ x &\longmapsto (x, U_{\text{old}}(x)\varphi_{\text{old}}(x)) \end{aligned}$$

So far, this is all just a fancier description of the same scalar field theory we did at the beginning, since we've only dealt with "trivial bundles" like  $\begin{array}{c} M \times F \\ \downarrow \\ M \end{array}$  like  $\begin{array}{c} M \times F \\ \downarrow \\ M \end{array}$  with  $(x, f)$  and  $x$ .

More general gauge theories will involve nontrivial bundles ones that are not isomorphic to ones like  $\begin{array}{c} M \times F \\ \downarrow \\ M \end{array}$ , as we'll see...