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In our bundle description of the N-component real scalar field φ, we think of φ as a section of a vector bundle:

$$\begin{tikzcd} M \times \mathbb{R}^N \arrow[d, "p"'] \\ M \arrow[u, bend right, "\varphi"'] \end{tikzcd}$$

and think of a gauge transformations $\mathcal{U}$ as a section of a bundle of groups:

$$\begin{tikzcd} M \times SO(N) \arrow[d] \\ M \arrow[u, bend right, "\mathcal{U}"'] \end{tikzcd}$$

The gauge transformation $\mathcal{U}$ acts on the field φ in an obvious way:

if $\quad \varphi(x) = (x, \tilde{\varphi}(x)) \quad$ and $\quad \mathcal{U}(x) = (x, \tilde{\mathcal{U}}(x))$

then we get a new scalar field $\mathcal{U}\varphi$ by:

$$(\mathcal{U}\varphi)(x) = (x, \tilde{\mathcal{U}}(x)\tilde{\varphi}(x))$$

This just rotates the fiber $\{x\} \times \mathbb{R}^N$ using the matrix $\tilde{\mathcal{U}}(x)$.

Any gauge transformation of this sort actually gives us a map of vector bundles:

$$\begin{tikzcd} (x,v) \arrow[r, mapsto] & (x, \tilde{U}(x)v) \\ M \times \mathbb{R}^N \arrow[r] \arrow[d, "p"'] & M \times \mathbb{R}^N \arrow[d, "p"] \\ M \arrow[r] & M \end{tikzcd}$$

This really is a map of <u>vector</u> bundles, since it's linear on fibers. It's also an isomorphism, with inverse $(x,v) \mapsto (x, \tilde{U}(x)^{-1}v)$. We usually collapse the 1_M and draw this as:

$$\begin{tikzcd} M \times \mathbb{R}^N \arrow[rr, "U"] \arrow[dr, "p"'] & & M \times \mathbb{R}^N \arrow[dl, "p"] \\ & M & \end{tikzcd}$$

where we use "U" to denote the bundle automorphism induced by the section U of $M \times SO(N)$. This abuse of notation is handy, since

$$\underset{\uparrow \text{the bundle automorphism}}{U}(\varphi(x)) = U(x, \tilde{\varphi}(x)) = (x, \tilde{U}(x)\tilde{\varphi}(x)) = \underset{\uparrow \text{the section of } M \times SO(N)}{(U\varphi)}(x)$$

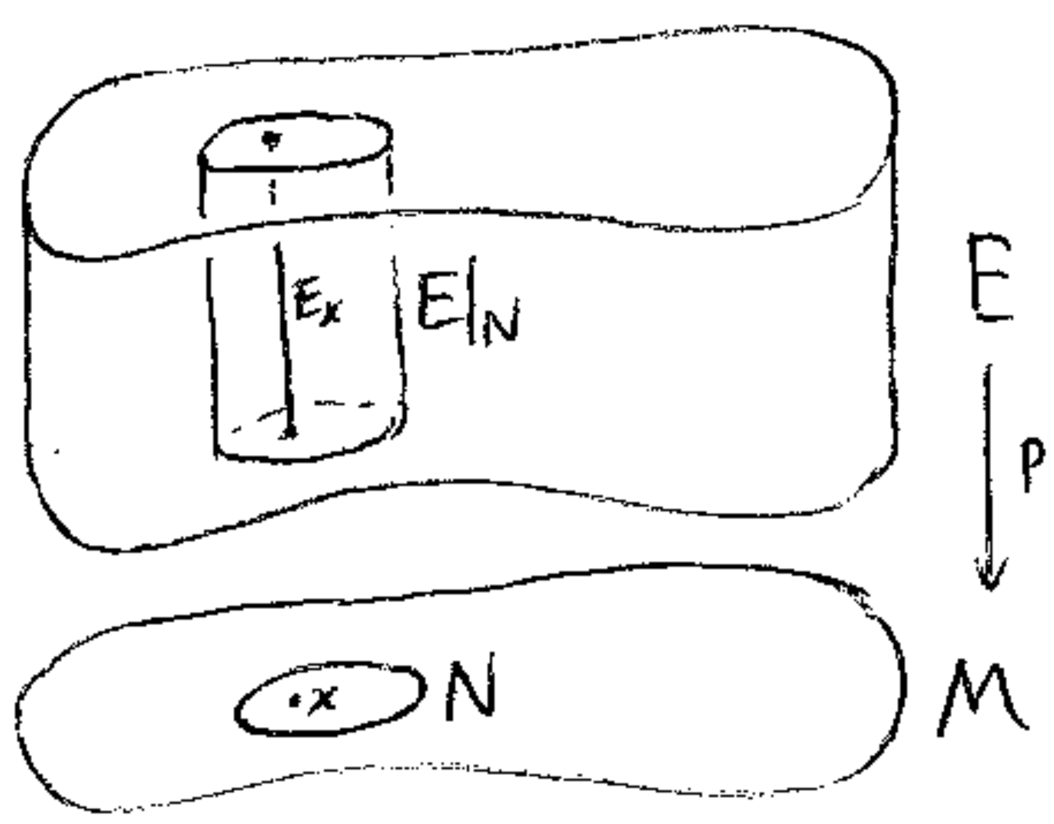
Notice that an automorphism of $M \times \mathbb{R}^N$ comes from some section of $M \times SO(N)$ in this way iff it preserves the inner product and orientation on each fiber. So, we're free to regard a gauge transformation as:

- a section of a certain bundle of groups, or
- an automorphism of a bundle, preserving certain structure.

In a more general gauge theory, "spacetime" is some manifold M and a typical "field" is a section of a vector bundle over M:

$$\begin{tikzcd} E \arrow[d, "p"'] \\ M \arrow[u, bend right, "\varphi"'] \end{tikzcd}$$

Here each fiber $E_x = p^{-1}\{x\}$ is a vector space, but E need not be trivial like $M \times \mathbb{R}^N$, or even "trivializable", meaning isomorphic to a trivial bundle. We do require E to be "locally trivial" — let's explain what this means. If $N \subseteq M$, E restricts to a bundle over N:



$$\begin{tikzcd} E|_N = \bigcup_{x \in N} E_x \arrow[d, "p"'] \\ N \end{tikzcd}$$

A vector bundle E is <u>locally trivial</u> if each $x \in M$ has an open nbhd N s.t. $E|_N$ is trivializable, i.e. there is an isomorphism of vector bundles

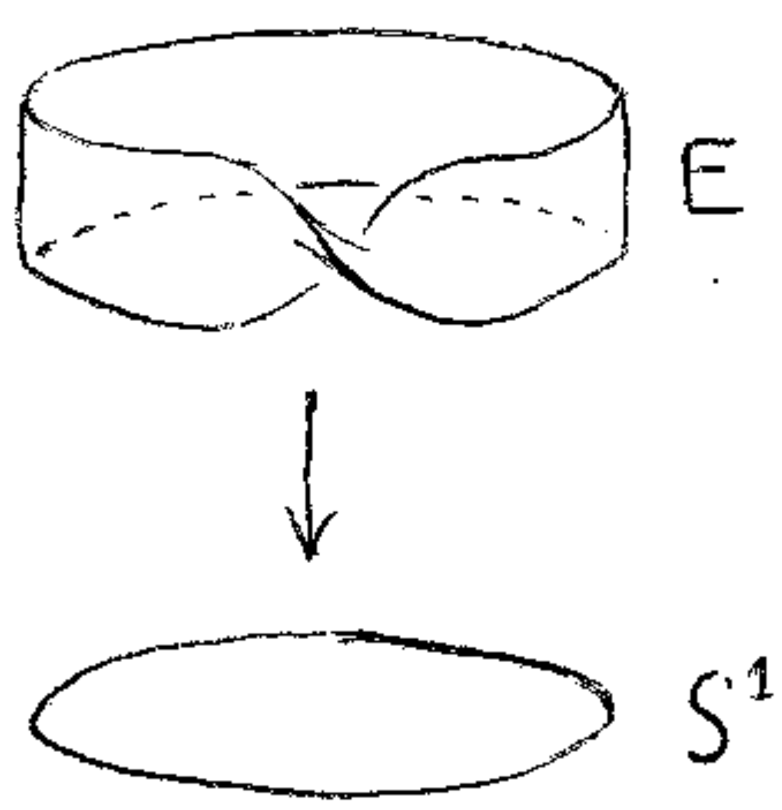
$$\begin{tikzcd} E|_N \arrow[rr, "\tau"] \arrow[dr] & & N \times V \arrow[dl] \\ & N & \end{tikzcd}$$

for some vector space V. In such a nbhd, we can use τ to pretend a section of E is really a function $N \to V$, but we can't do this globally unless E is trivializable.

The map τ is called a <u>local trivialization</u> of E over $N \subseteq M$. If every fiber looks like V, E is called a locally trivial vector bundle with <u>typical fiber</u> V, or <u>standard fiber</u> V. (Obviously, all this stuff generalizes to other kinds of bundles, too.)

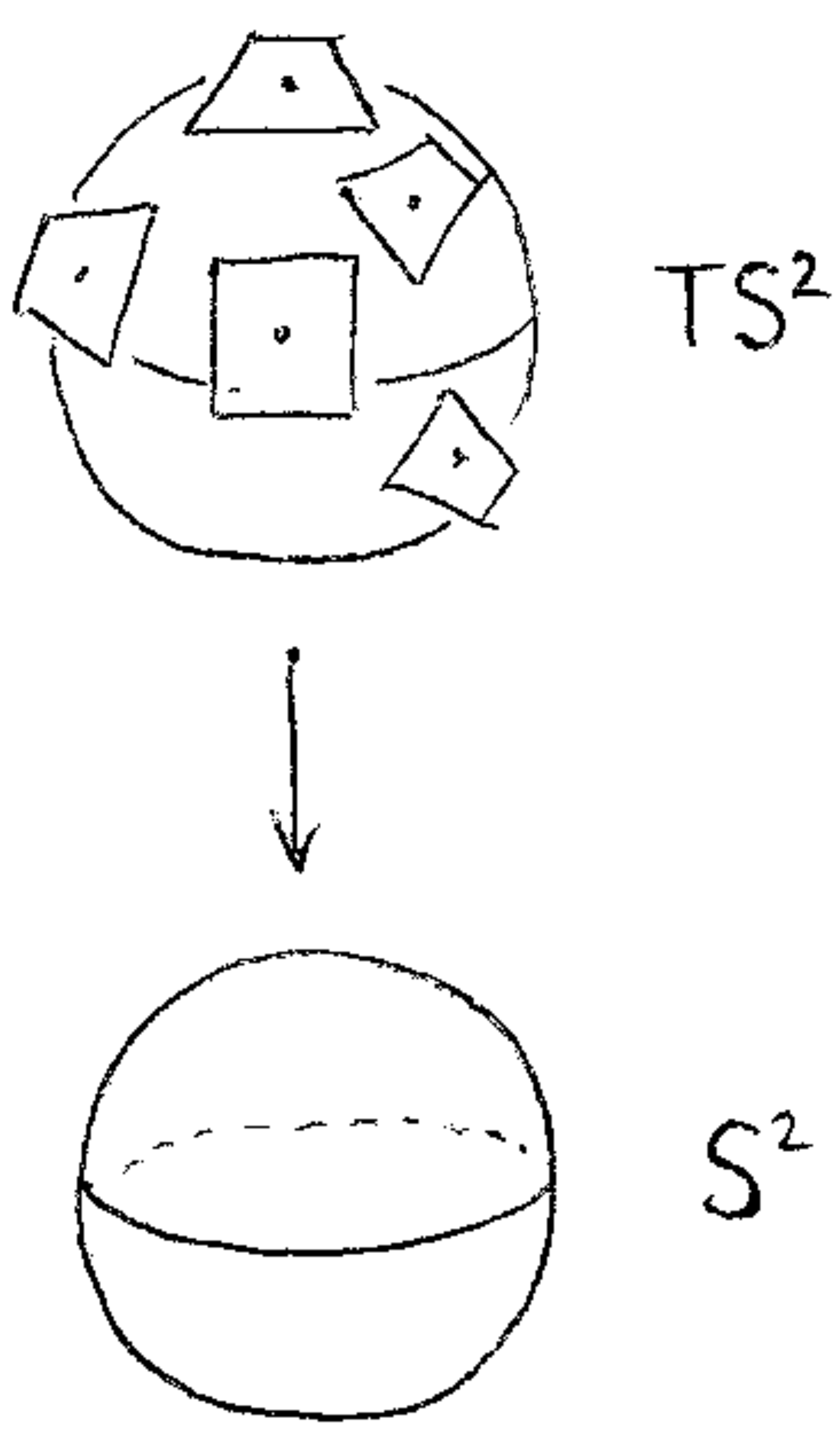
<u>Examples of nontrivial vector bundles</u>:

1. The Möbius bundle



Each fiber looks like $\mathbb{R}$, but $E \not\cong S^1 \times \mathbb{R}$ because of the twist. However, if $N \subsetneq S^1$ is any proper open set, $E|_N$ is trivializable, so the Möbius bundle on S^1 is locally trivial.

2. Tangent bundle of the sphere



$TS^2 = \bigcup_x T_x S^2$

= union of all tangent planes to S^2

This is locally trivial, but not trivializable

(Proof: if it were trivializable:

$$\begin{tikzcd} TS^2 \arrow[rr, "\alpha"] \arrow[dr] & & S^2 \times \mathbb{R}^2 \arrow[dl] \\ & S^2 \arrow[ur, bend right, "v"'] & \end{tikzcd}$$

section given by $v(x) = (x, (1,0))$

then $\alpha^{-1} \circ v$ would be a nonvanishing section of TS^2, i.e. a nonvanishing vector field on S^2, contradicting the coconut theorem.)

Next, if the field φ is a section of $\begin{smallmatrix} E \\ \downarrow \\ M \end{smallmatrix}$, what's a gauge transformation? Following the hint from the trivial case, it should be a bundle automorphism
$$\begin{tikzcd}[column sep=small] E \arrow[rr] \arrow[dr] & & E \arrow[dl] \\ & M & \end{tikzcd}$$
of the appropriate type — i.e. preserving the relevant structure on fibers. In the N-component scalar field example, the bundle $M \times \mathbb{R}^N$ was more than just a vector bundle: it was a bundle of inner product spaces! (Our Lagrangian used the dot product in $\mathbb{R}^N$!) So, to generalize this theory, φ should be a section of a locally trivial bundle of inner product spaces

$$\begin{tikzcd} E \arrow[d] \\ M \arrow[u, bend right, "\varphi"'] \end{tikzcd} \qquad \begin{tikzcd} E|_N \arrow[rr, "\cong", "\tau"'] \arrow[dr] & & N \times V \arrow[dl] \\ & N & \end{tikzcd}$$

V an inner product space
$\tau : E_x \longrightarrow V$ iso. of inner product spaces

(actually, we also implicitly assumed an orientation on each copy of $\mathbb{R}^N$, since we used $SO(N)$ rather than $O(N)$, so we might want a locally trivial bundle of oriented inner product spaces.)

A gauge transformation should be an automorphism of this bundle — a bundle isomorphism that's linear and preserves the inner product (and orientation) on each fiber.

We are now ready to write down the Lagrangian for an "N-component real scalar field theory", in our more general setting:

$$\mathcal{L} = -g^{\mu\nu}\langle D_\mu\varphi, D_\nu\varphi\rangle - V(\langle\varphi,\varphi\rangle)$$

Here, spacetime is a Lorentzian manifold M with metric $g_{\mu\nu}$

φ is a section of a bundle of inner product spaces $\begin{array}{c} E \\ \downarrow \\ M \end{array}$ φ

$\langle\ ,\ \rangle$ is a smoothly varying inner product on each fiber

(really $\langle\varphi,\varphi\rangle$ means $\langle\varphi(x),\varphi(x)\rangle_{E_x}$)

There's just one thing we haven't defined yet — what's D? It's a "connection" on the vector bundle E. A connection is just a natural way of differentiating sections to get new sections. We won't define connections carefully yet, but let's note some important properties:

- There's no canonical choice of D — there are lots of inequivalent ways to differentiate sections of a vector bundle. So, $\mathcal{L}$ depends on the connection used:

$$\mathcal{L} = \mathcal{L}[\varphi, D]$$

- For a given connection D, we can take a section φ of $\begin{smallmatrix} E \\ \downarrow \\ M \end{smallmatrix}$ and a vector field w on M and get a new section $D_w\varphi$:

$$\begin{tikzcd} E \arrow[d, shift right] \\ M \arrow[u, shift right, bend right, "D_w\varphi"'] \end{tikzcd}$$

called the <u>covariant derivative</u> of φ in the direction of w.

- In the Lagrangian, "$D_\mu\varphi$" only makes sense in a local chart — it stands for $D_{\partial_\mu}\varphi$, where ∂_μ is the local vector field in the x^μ coordinate direction. However, $g^{\mu\nu}\langle D_\mu\varphi, D_\nu\varphi\rangle$ is a function on M that doesn't depend on the choice of coordinates, so it makes sense globally. (we could just write $g\langle D\varphi, D\varphi\rangle$)

- Gauge invariance: A gauge transformation $\begin{tikzcd} E \arrow[rr, "U", "\sim"'] \arrow[dr] & & E \arrow[dl] \\ & M & \end{tikzcd}$ acts on sections by composition:

$$\varphi \longmapsto U\varphi \qquad \begin{tikzcd} E \arrow[rr, "U"] \arrow[dr] & & E \arrow[dl] \\ & M \arrow[ul, bend left, "\varphi"] \arrow[ur, "U\varphi"'] & \end{tikzcd}$$

Since $D_\mu\varphi$ is also a (local) section, the gauge transformation must act in the same way:

$$D_\mu\varphi \longmapsto U D_\mu \varphi$$

Since U preserves the metric on each fiber by definition, the Lagrangian is invariant.

- In local coordinates, we can always write

$$D_\mu \varphi = \partial_\mu \varphi + A_\mu \varphi$$

as in our earlier description.

<u>Exercise</u>: Show that, applying a gauge transformation U, the new section $UD_\mu\varphi$ can be written, in the same coordinates, as $\partial_\mu\varphi + \tilde{A}_\mu\varphi$ where

$$\tilde{A}_\mu = UA_\mu U^{-1} + U(\partial_\mu U^{-1})$$

so that the above is consistent with our description of the trivial bundle case.

(I've told you everything you need to do this exercise, but you can look up the definition of connection on a vector bundle, e.g. in Baez & Muniain, <u>Gauge Fields, Knots, and Gravity</u>)