**Gauge Theory and Gravity**

Lecture 3: Principal Bundles
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## Bundles

Recall that a bundle is just a smooth map

$$\begin{array}{c} E \\ {\scriptstyle p}\big\downarrow \\ M \end{array}$$

Usually each fiber $E_x = p^{-1}\{x\}$ has the structure of some 'gadget' (vector space, group, etc.); when this is the case, and when all of the gadget operations vary smoothly from fiber to fiber, $p$ is a 'bundle of gadgets'.

A bundle map is a pair of maps $\alpha, \tilde{\alpha}$ making the diagram

$$\begin{array}{ccc} E & \xrightarrow{\tilde{\alpha}} & E' \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p'} \\ M & \xrightarrow{\alpha} & M' \end{array}$$

commute. For maps between 'bundles of gadgets', we require that for each $x$, $\tilde{\alpha}\colon E_x \to E_{\alpha(x)}$ is a 'gadgetomorphism' (linear map, in the case of vector bundles, group homomorphism in the case of bundles of groups, etc.). A **gauge transformation** of a 'bundle of gadgets' $E \to M$ is a just a 'gadget-bundle automorphism' (with the identity map on the base space):

$$\begin{array}{ccc} E & \longrightarrow & E \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & M & \end{array}$$

## $G$-bundles

Let $G$ be a Lie group.

A **right $G$-space** is a manifold $Y$ with a smooth right $G$-action

$$\begin{array}{rcl} E & \longrightarrow & E \\ (y,g) & \longmapsto & yg \end{array}$$

The natural kinds of maps between $G$-spaces are **$G$-equivariant** maps: smooth maps $f\colon E \to E$ such that $f(yg) = f(y)g$ for all $y \in E$ and $g \in G$.

A **$G$-bundle** is a locally trivial bundle $E \to M$ such that $E$ is a $G$-space and the action of $G$ preserves fibers, i.e. $y \in E_x$ implies $yg \in E_x$ for every $g$. Note this makes each fiber into a $G$-space, so a $G$ bundle is a locally trivial bundle of $G$-spaces.

If $E$ is a $G$-bundle, the action of $G$ gives, for each element $g \in G$, a gauge transformation:

$$\begin{array}{ccc} E & \xrightarrow{g} & E \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & M & \end{array}$$

which we can think of a "global gauge transformation". (Note this really is a map of $G$-bundles, since it preserves fibers and is also $G$-equivariant, thanks to the equation $(yg)g' = y(gg')$, which is part of the definition of an action). There, are of course, more general gauge transformations than these—one for every invertible $G$-equivariant map $E \to E$.

The most important kind of $G$-bundles are 'principal bundles'. A **principal $G$-bundle** is a $G$-bundle for which, whenever $y, y' \in E_x$, there exits a unique $g$ in $G$ such that $yg = y'$. In a principal bundle, note that if we pick $y_o \in E_x$, the map

$$\begin{array}{ccc} G & \longrightarrow & E_x \\ g & \longmapsto & y_o g \end{array}$$

is an *isomorphism* of $G$-spaces.

So: a principal $G$ bundle is a locally trivial bundle of $G$-spaces, where every fiber is isomorphic, as a right-$G$-space, to $G$.

## Example: The Frame Bundle

Consider the tangent bundle

$$\begin{array}{c} TM \\ \downarrow \\ M \end{array}$$

where $M$ is a Riemannian $n$-dimensional manifold. This is a locally trivial bundle of inner product spaces; every fiber $T_xM$ is isomorphic (as an inner product space) to $\mathbb{R}^n$. The isomorphism is, of course, not canonical; a choice of isomorphism

$$f : \mathbb{R}^n \to T_xM$$

is called an **(orthonormal) frame** at $x$. Denote by $F_x$ the space of all frames at $x$. These can be assembled into a bundle $FM$—the **frame bundle** of $M$. Any two frames are related by a *unique* orthogonal transformation of $\mathbb{R}^n$:

$$\begin{array}{ccc} \mathbb{R}^n & \overset{f}{\searrow} & \\ {\scriptstyle g}\downarrow & & T_xM \\ \mathbb{R}^n & \underset{f'}{\nearrow} & \end{array}$$

This makes the frame bundle into a *principal* $O(n)$-bundle over $M$, where the action of $g \in O(n)$ is simply by composition:

$$\begin{array}{ccc} F_xM & \longrightarrow & F_xM \\ (f, g) & \longmapsto & f \circ g \end{array}$$

(There are other versions of the frame bundle: If $M$ is an *oriented* Riemannian manifold, we can take only frames compatible with the orientation, and the frame bundle is a principal $SO(n)$-bundle; If $M$ is a smooth manifold without metric, a frame is just an isomorphism as vector spaces, and the frame bundle is a principal $GL(n, \mathbb{R})$ -bundle.)

## Principal bundles as generalized frame bundles

It's good to think of *any* principal bundle as a bundle of "generalized frames" for some kind of bundle. If $E \to M$ is some locally trivial 'gadget-bundle', whose fibers are all isomorphic to some 'gadget' $V$, define a **generalized frame** at $x \in M$ to be a choice of isomorphism

$$f: V \to E_x$$

**Exercise:** We can form a bundle $P \to M$ of generalized frames, and this is a principal $G$-bundle where $G$ is the group of 'gadget automorphisms' of the standard fiber $V$.

(We don't have a formal definition of gadget, but this works in obvious examples—group, vector space, algebra, Lie algebra, $G$-space, etc. One can prove this in specific cases, or state and prove more general theorems that include wide classes of gadgets.)

This shows that all other locally trivial bundles of gadgets (of whatever type) are related to some principal bundle in the same way that the tangent bundle is related to the frame bundle. Next time we'll see how to reconstruct the bundle of gadgets from its principal bundle of generalized frames. So, most of gauge theory can be reduced to the study of *principal* bundles.