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Associated Bundles

Given any locally trivial "bundle of gadgets" $\begin{array}{c} E \\ \downarrow \\ M \end{array}$ with standard fiber F (some "gadget"), we've seen how to get a principal G bundle, where $G = \mathrm{Aut}(F)$ is the group of "gadget automorphisms" of F. Namely we let

$$P_x = \{ \text{gadget isomorphisms } f : F \longrightarrow E_x \}$$

be the space of "generalized frames" at x, and glue these together smoothly to get a bundle:

$$\begin{tikzcd} P = \bigcup_{x \in M} P_x \arrow[d] \\ M \end{tikzcd}$$

This is a <u>principal</u> bundle because each P_x is isomorphic — as a right G-space, not as a group — to G itself:

$$\begin{tikzcd} F \arrow[dr, "f"] \arrow[d, dashed, "\exists! g"'] & \\ F \arrow[r, "f'"'] & E_x \end{tikzcd} \qquad \begin{array}{l} \forall f, f' \in P_x \\ \exists!\, g \in G \text{ s.t. } f'g = f. \end{array}$$

Today, we'll see how to reconstruct the bundle E from its principal bundle P of generalized frames and its standard fiber F — and also get lots of new bundles.

EXAMPLE: Let Σ be a surface, $F\Sigma$ its frame bundle, where a frame is any linear isomorphism

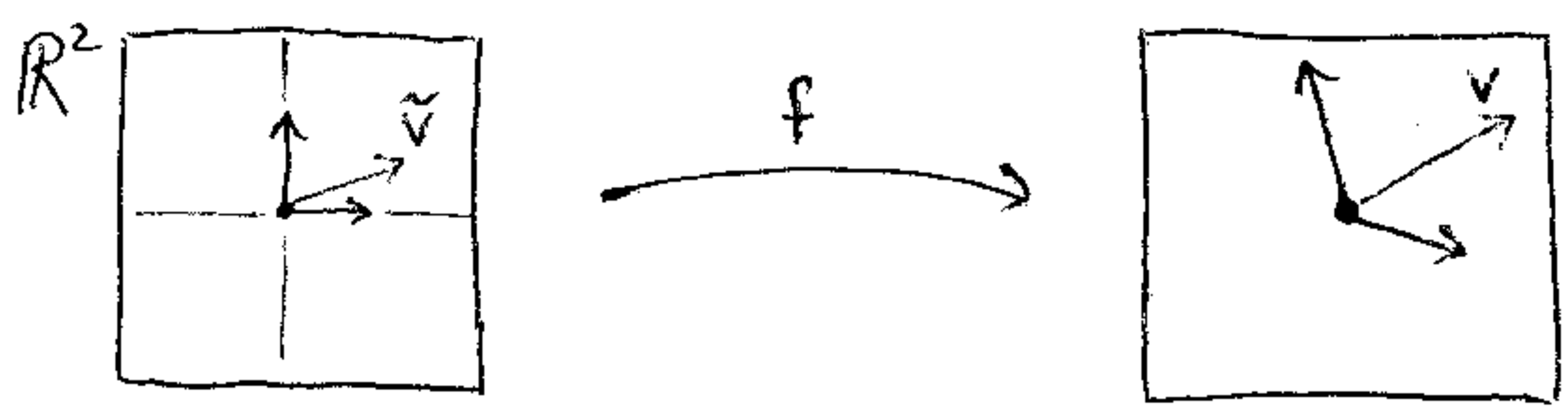
$$f : \mathbb{R}^2 \longrightarrow T_x\Sigma .$$

Any two frames at x are related by a unique elt. of

$$GL(2,\mathbb{R}) = \{\text{linear automorphisms of } \mathbb{R}^2\}$$

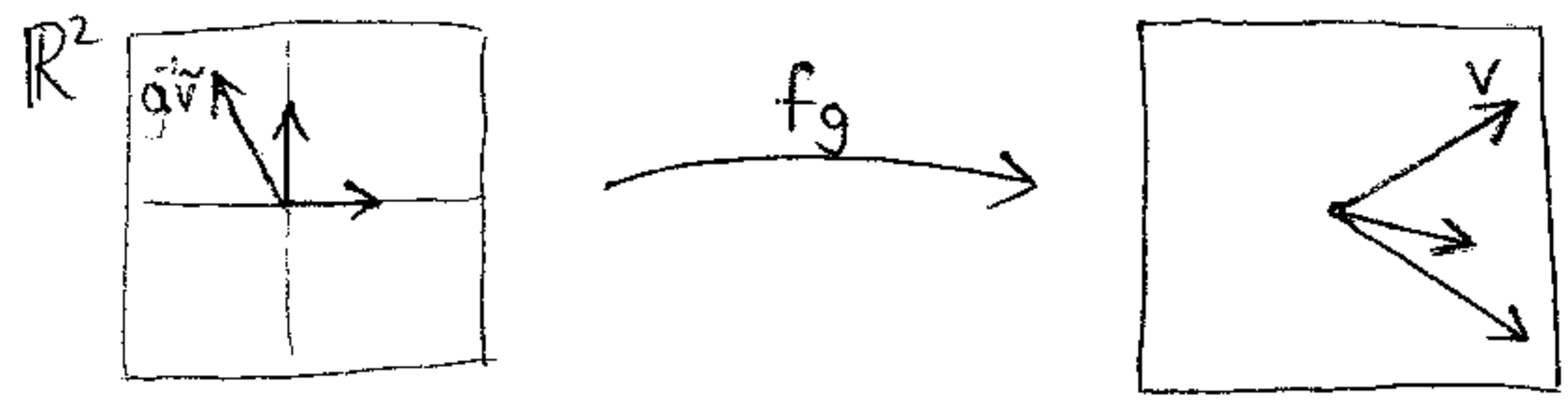
so $F\Sigma$ is a principal $GL(2,\mathbb{R})$-bundle. Now consider the reverse process: suppose we're given some principal $GL(2,\mathbb{R})$-bundle $\begin{smallmatrix} P \\ \downarrow \\ \Sigma \end{smallmatrix}$ and we want to construct a 2d vector bundle $\begin{smallmatrix} E \\ \downarrow \\ \Sigma \end{smallmatrix}$ that has P as its "bundle of frames."

Each E_x should be a 2d vector space; if an elt. $f \in P_x$ is to be thought of as a "frame" at x, it should give a basis of E_x, as the image of the standard basis:



An elt. $v \in E_x$ can then be specified by a "frame" $f \in P_x$ together with an elt. $\tilde{v} \in \mathbb{R}^2$ — the "components of v in the frame f". <u>BUT</u>, the pair $(f, \tilde{v})$ is not unique! ...

If I change f to fg, $g \in GL(2,\mathbb{R})$, then the same vector $v \in E_x$ should now have components $g^{-1}v \in \mathbb{R}^2$:



So: we could say E_x consists of pairs $(f, \tilde{v}) \in P_x \times \mathbb{R}^2$ modulo identification of $(f, \tilde{v})$ with $(fg, g^{-1}\tilde{v})$ for all $g \in GL(2,\mathbb{R})$.

This example motivates the following definition:

First, let G be a Lie group, P a principal G-bundle over a manifold M:

$$\begin{array}{c} P \\ \downarrow \pi \\ M \end{array}$$

and let V be a (left) G-space. Define an equivalence relation on $P \times V$ by

$$(p,v) \sim (p',v') \iff p' = pg,\ v' = g^{-1}v \quad \exists g \in G;$$

in other words:

$$(pg, v) \sim (p, gv) \quad \forall g \in G,\ p \in P,\ v \in V.$$

Then let

$$P \times_G V := P \times V / \sim$$

be the space of equivalence classes, and denote the class of $(p,v) \in P \times V$ by $[p,v] \in P \times_G V$.

<u>Def:</u> If $\begin{array}{c} P \\ \downarrow\pi \\ M \end{array}$ is a principal G-bundle and V is a G-space, then the bundle

$$\begin{array}{ccc} P \times_G V & & [p,v] \\ \downarrow & & \downarrow \\ M & & \pi(p) \end{array}$$

is called the bundle <u>associated to</u> P with standard fiber V.

(Note: this definition actually requires a few little <u>theorems</u> before it makes sense, e.g.:

- $P \times_G V$ is smooth mfld.
- the projection $[p,v] \longmapsto \pi(p)$ is smooth
- $P \times_G V$ really is locally trivial — locally iso. to a trivial bundle — so the phrase "with standard fiber" has its standard meaning.

Most of this is easy, but we won't worry about it.)

Back to our example, we really do get

$$F\Sigma \times_{GL(2,\mathbb{R})} \mathbb{R}^2 \cong T\Sigma$$

So "taking the associated bundle" seems to be the inverse of "taking the bundle of frames". This is true more generally...

<u>Thm</u>: Suppose $\begin{smallmatrix}E\\ \downarrow\\ M\end{smallmatrix}$ is a "bundle of gadgets" with standard fiber V and let $\begin{smallmatrix}P\\ \downarrow\\ M\end{smallmatrix}$ be the bundle of generalized frames $f: V \to E_x$ — a principal G bundle, $G = \mathrm{Aut}(V)$. Then there is a canonical isomorphism of bundles of gadgets:

$$P \times_G V \cong E$$

<u>Pf</u>: Exercise! (You can ignore technical details such as smoothness conditions — just construct an explicit "gadget-bundle isomorphism".) ∎

Examples of Associated Bundles

Once we have a principal G-bundle, we can construct tons of new bundles — one for every G-space V we can imagine. Better yet, if V has the structure of some gadget and the action of G is as gadget automorphisms, the associated bundle will be a bundle of gadgets.

<u>Examples</u>: (for G and a principal G-bundle $\begin{smallmatrix}P\\ \downarrow\\ M\end{smallmatrix}$)

1) If G acts as linear transformations of the <u>vector space</u> V (i.e. G has a <u>representation</u> on V) then we get a <u>vector bundle</u> $P \times_G V$.

(e.g. $G = GL(n, \mathbb{R})$, $P = FM$, $V = \mathbb{R}^n \rightsquigarrow P \times_G V \cong TM$)

2) If V is an inner product space and G has a rep. on V that preserves the inner product, then we get a <u>bundle of inner product spaces</u> $P \times_G V$.

(e.g. $G = O(n)$, $P = FM$ = the <u>orthonormal</u> frame bundle, $V = \mathbb{R}^n \rightsquigarrow P \times_G V \cong (TM, g)$ = tangent bundle with metric)

3) If $V = G'$ is a group, and $G \subseteq G'$, then G acts on G' by multiplication

$$\begin{aligned} L : G \times G' &\longrightarrow G' \\ (g, g') &\longmapsto gg' =: L_g g' \end{aligned}$$

and the associated bundle

$$P \times_G G'$$

is a <u>principal G'-bundle</u> — <u>Not</u> a bundle of groups! Why? Because left multiplication by $g \in G$ is <u>not</u> an automorphism of G' as a group. But it <u>is</u> an automorphism of G' as a right G'-space.

<u>Exercise</u>: Show that $P \times_G G'$ really is a principal G' bundle. If $G = G'$, show $P \times_G G \cong P$ as principal bundles.

4) Again taking $G \subseteq G'$, G also acts on G' by conjugation:

$$\begin{aligned} AD : G \times G' &\longrightarrow G' \\ (g, g') &\longmapsto g g' g^{-1} =: AD(g)g' \end{aligned}$$

For each g, $AD(g) : G' \longrightarrow G'$ is a group automorphism, so the associated bundle is a bundle of groups, not just torsors. To distinguish it from example 3, we denote the associated bundle

$$P \times_{AD_G} G'.$$

Example 4 is very important, especially in the case $G = G'$:

Thm: Let P be a principal G-bundle, $P \times_{AD} G$ the bundle of groups associated to P via the action of G on itself by conjugation. Then there is a group isomorphism:

Gauge transformations

$$\begin{tikzcd} P \arrow[rr, "\alpha"] \arrow[dr] & & P \arrow[dl] \\ & M & \end{tikzcd}$$

(w. composition of principal bundle maps)

$\cong$

Sections of associated group-bundle

$$\begin{tikzcd} P \times_{AD} G \arrow[d] \\ M \arrow[u, bend right, "\alpha"'] \end{tikzcd}$$

(w. pointwise multiplication of sections)

Pf: Homework! (This is the version for nontrivial bundles of our earlier observation (Lecture 2) that gauge transformations may be viewed either as bundle automorphisms or a sections of a certain group bundle.)