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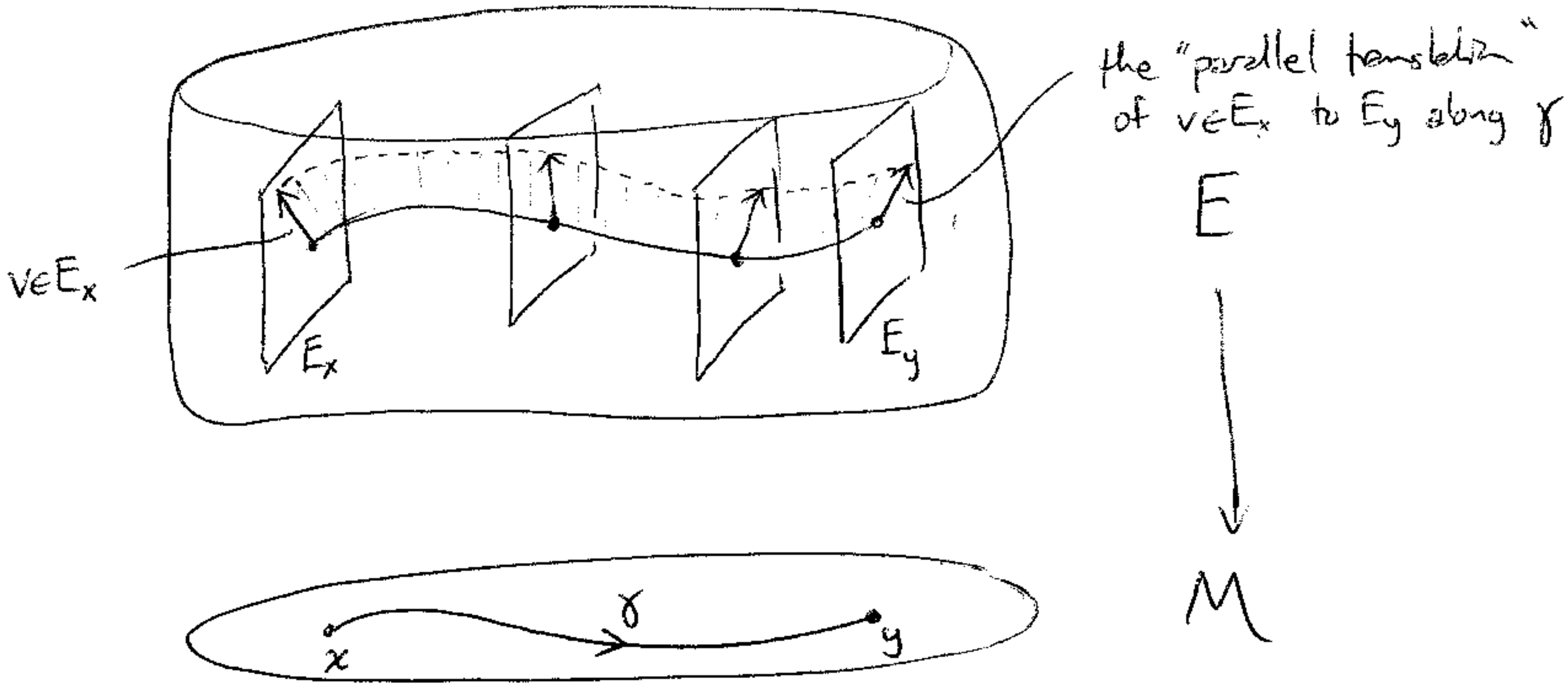
Connections & Holonomy

A "connection" on a bundle is what lets us do "parallel translation", comparing elements of different fibers by dragging them along paths in the base space, in a specified way. E.g. in a vector bundle:



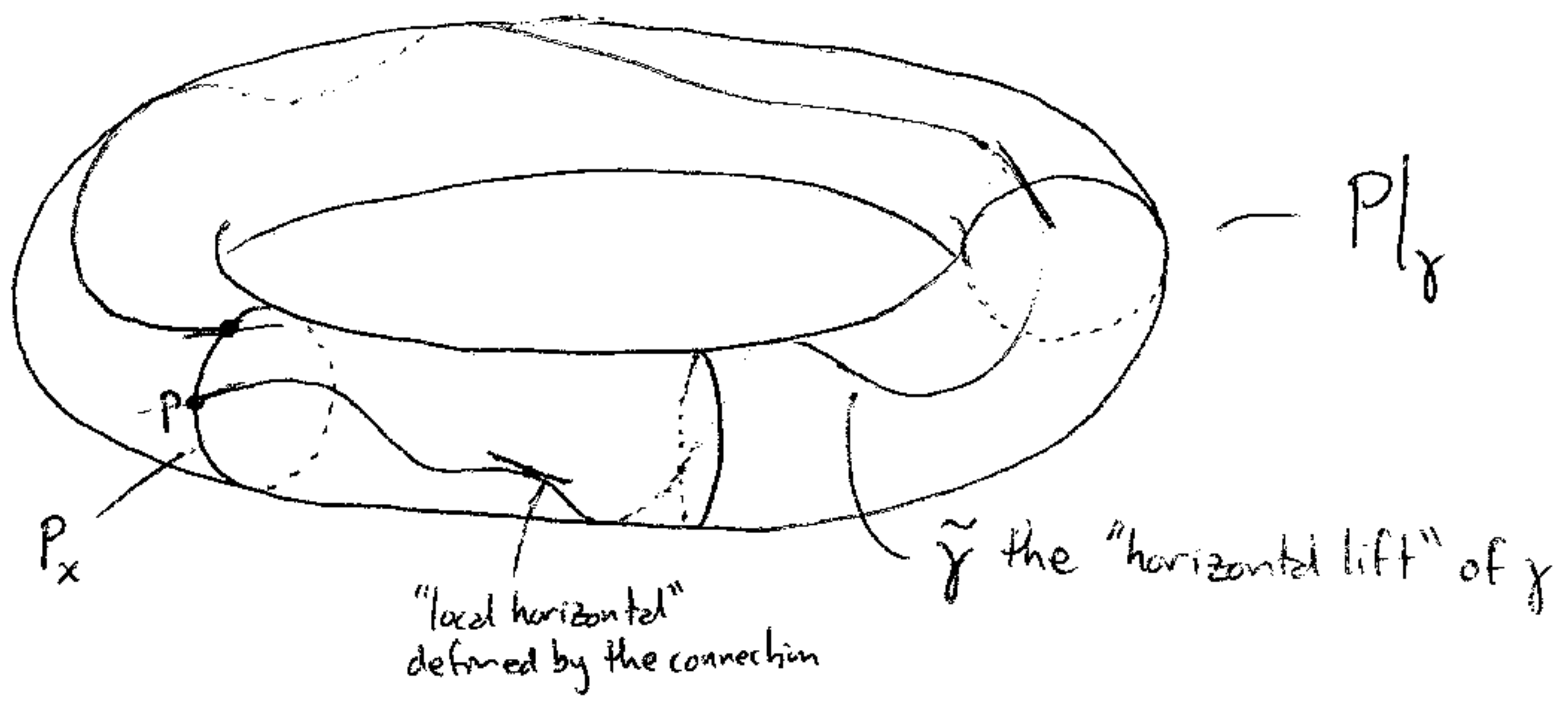
Connections on vector bundles are important for differentiating sections, since the naive derivative formula

$$\text{"} \lim_{t' \to 0} \frac{\sigma(\gamma(t+t')) - \sigma(\gamma(t))}{t'} \text{"}$$

doesn't make sense — we can't subtract elements of different fibers! To make sense of it, we have to parallel translate one of these elements over to the fiber the other element lives in and <u>then</u> subtract. This is what "covariant differentiation" amounts to.

But, we're getting ahead of ourselves; first we need to understand connections on arbitrary bundles...

The most fundamental kind is a connection on a <u>principal</u> G-bundle P; connections on all its associated bundles $P \times_G V$ can be derived from a connection on P. Parallel translation in P, say around a <u>loop</u> in M, looks like this:



Formally, we have the following definition:

<u>Def</u> – A connection (more precisely, an Ehresmann connection) on a principal G-bundle $\begin{matrix} P \\ \downarrow \pi \\ M \end{matrix}$ is a collection of linear maps, one for each $p \in P$,

$$A_p : T_{\pi(p)} M \longrightarrow T_p P$$

s.t.:

0) A_p varies smoothly with p

1) $d\pi \circ A_p = 1_{T_{\pi(p)} M}$

2) $A_{pg} = dR_g \circ A_p$

Let's explain this definition and how it prosribes a way to do parallel translation:

1)
$$\begin{tikzcd} P \arrow[d, "\pi"] \\ M \end{tikzcd} \quad \text{gives} \quad \begin{tikzcd} T_pP \arrow[d, "d\pi"'] \\ T_{\pi(p)}M \arrow[u, bend right, "A_p"'] \end{tikzcd}$$

We call $A_p(v)$ the "horizontal lift" of $v \in T_{\pi(p)}M$ to T_pP.

So $d\pi \circ A_p = 1_{T_{\pi(p)}M}$ just says the horizontal lift of a vector v projects back down to v. In particular, this means A_p is one-to-one, so $\operatorname{Ran} A_p \subset T_pP$ is an n-dimensional subspace called the "horizontal subspace" of T_pP $(n = \dim M)$

2) Since P is a <u>principal</u> G-bundle, $\forall g \in G$ we get a gauge transformation

$$\begin{tikzcd} P \arrow[rr, "R_g"] \arrow[dr] & & P \arrow[dl] \\ & M & \end{tikzcd}$$

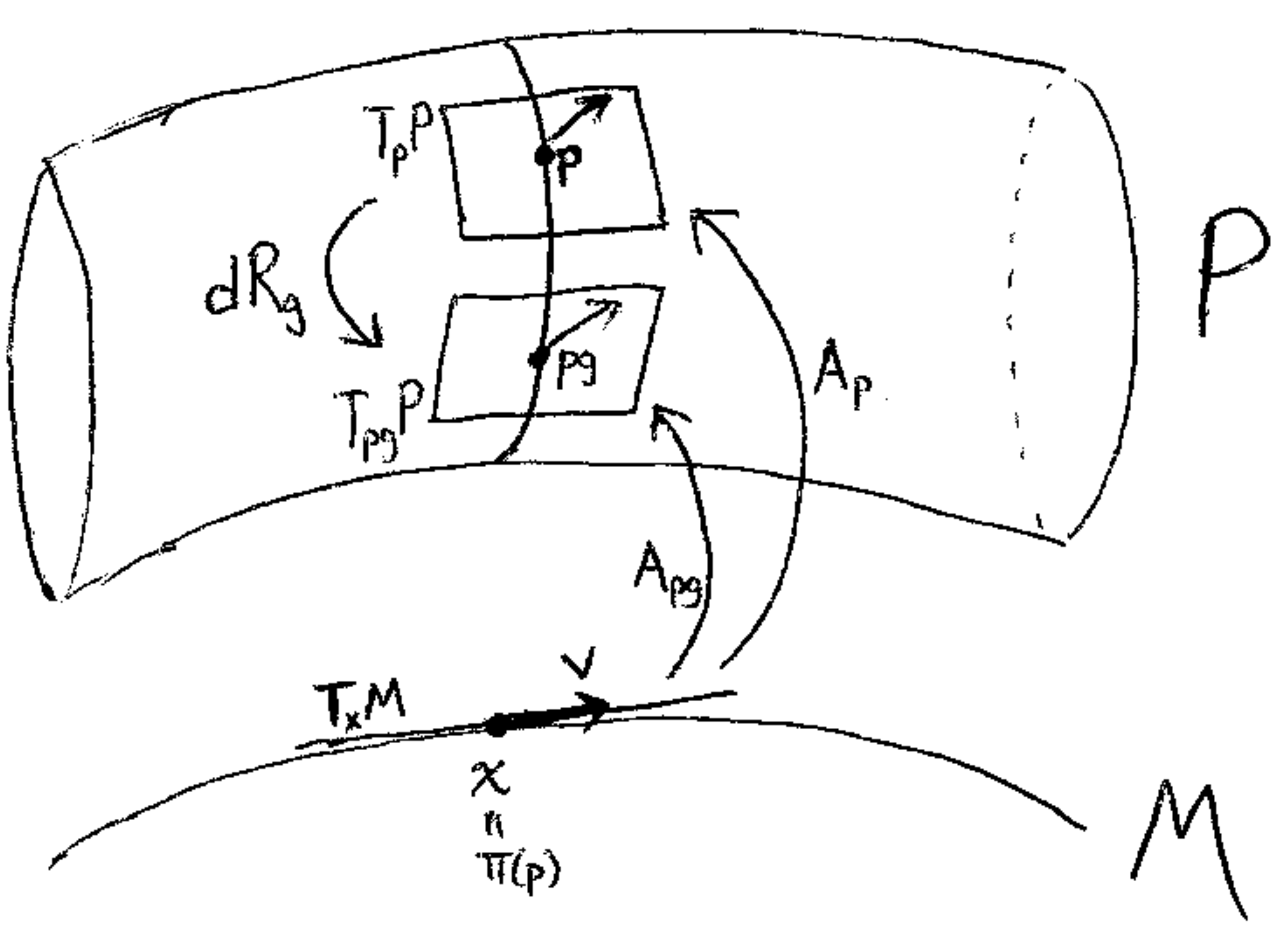
which acts on each fiber via the right action of G on P:
$$R_g(p) = pg.$$

The condition $A_{pg} = dR_g \circ A_p$ says

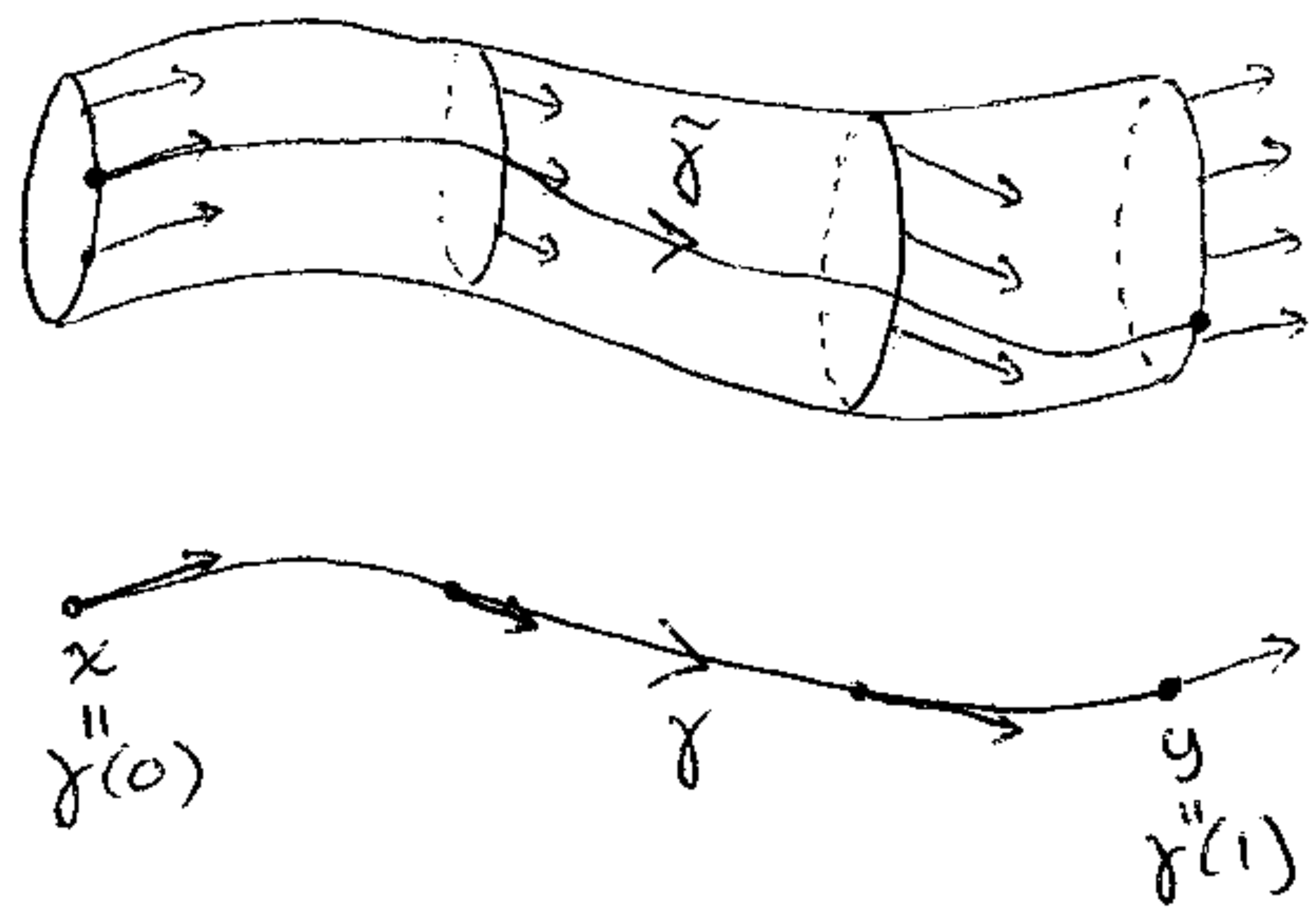
$$\begin{tikzcd} T_{\pi(p)}M \arrow[r, "A_p"] \arrow[rr, bend right, "A_{pg}"'] & T_pP \arrow[r, "dR_g"] & T_{pg}P \end{tikzcd}$$

commutes.

This means horizontal lifts in a given fiber are all "parallel" — they're related by some R_g:



Parallel transport in P (with respect to the connection A) is accomplished by taking <u>all</u> horizontal lifts of velocity vectors along a path and taking integral curves of the resulting vector field on $P|_\gamma$:



In other words, to find the horizontally lifted path $\tilde{\gamma}:[0,1]\to P$, we have to solve the differential equation

$$\tilde{\gamma}'(t) = A_{\tilde{\gamma}(t)}(\gamma'(t))$$

velocity of the horizontal lift of γ = horizontal lift of the velocity of γ

The basic existence and uniqueness thm for 1st order ODE can be used to show this has a unique solution satisfying an initial condition of the form

$$\tilde{\gamma}(0) = p$$

(with $\pi(p) = \gamma(0)$). How are the horizontal lifts $\tilde{\gamma}$ for different initial conditions related? ...

Let $\tilde{\gamma}_p : [0,1] \to P$ be the horizontal lift of γ starting at $p \in P$, i.e. $\tilde{\gamma}_p(0) = p$. Given $g \in G$, we get a new lift of γ by composing with the "global gauge transformation" $R_g : P \to P$:

$$R_g \tilde{\gamma}_p : [0,1] \longrightarrow P$$

Let's show this also satisfies the ODE for a horizontal lift:

LHS: $\quad (R_g \tilde{\gamma}_p)'(t) = dR_g(\tilde{\gamma}_p'(t)) \quad$ (by chain rule)

RHS: $\quad A_{R_g \tilde{\gamma}_p(t)}(\gamma'(t)) = A_{\tilde{\gamma}_p(t) g}(\gamma'(t)) \quad$ (def. of R_g)

$$= dR_g A_{\tilde{\gamma}_p(t)}(\gamma'(t)) \quad \text{(property 2 in def of connection)}$$

$$= dR_g(\tilde{\gamma}_p'(t)) \quad (\tilde{\gamma}_p \text{ is a horizontal lift})$$

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Since $R_g \tilde{\gamma}_p(0) = pg$, by uniqueness of horizontal lifts, this gives us

$$\tilde{\gamma}_{pg}(t) = R_g \tilde{\gamma}_p(t) = \tilde{\gamma}_p(t) g .$$

Now for each path $\gamma : [0,1] \to M$ we get a map (for a given connection):

$$\begin{aligned} \mathrm{hol}(\gamma) : P_{\gamma(0)} &\longrightarrow P_{\gamma(1)} \\ p &\longmapsto \tilde{\gamma}_p(1) \end{aligned}$$

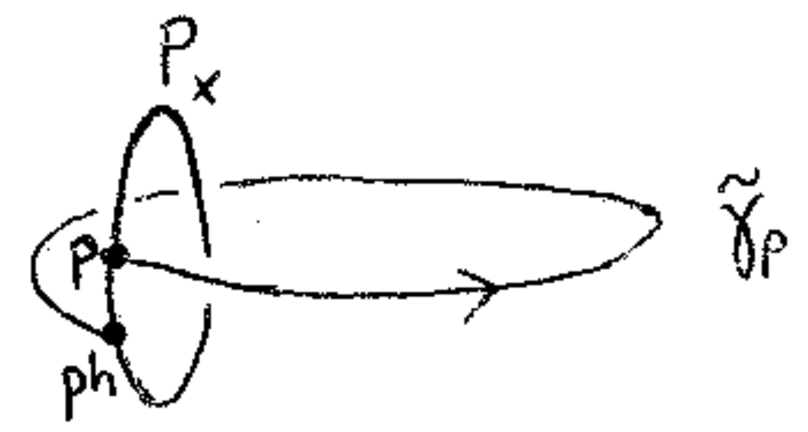
called the "holonomy" or "parallel transport" along γ. Note

$$\mathrm{hol}(\gamma)(pg) = \tilde{\gamma}_{pg}(1) = \tilde{\gamma}_p(1) g = \mathrm{hol}(\gamma)(p) g$$

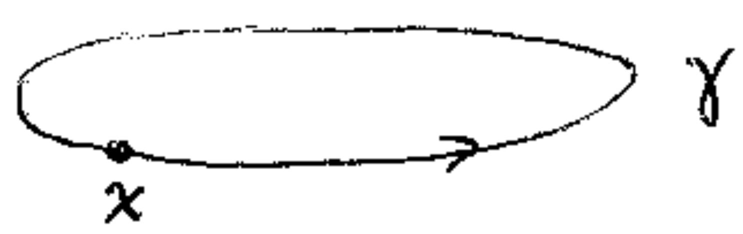
so $\mathrm{hol}(\gamma)$ is a map of G-spaces.

<u>Exercises</u>: 1) $hol(\gamma)$ is an <u>isomorphism</u> of G-spaces (hint: what's its inverse? 2) $hol(\gamma)$ doesn't depend on the parameterization of γ.

Holonomy is particularly interesting around a loop:



$hol(\gamma)(p) = ph \in P_x \quad \exists!\, h \in G$



$\gamma(0) = \gamma(1) = x$

Since P is a <u>principal</u> G bundle, and $hol(\gamma)$ is an automorphism of P_x, given $p \in P_x$ there's a <u>unique</u> $h \in G$ s.t. $hol(\gamma)(p) = ph$. So, we could call h "the holonomy" of the loop. But this isn't quite right, since if $hol(\gamma)(p) = ph$ then

$$\begin{aligned} hol(\gamma)(pg) &= hol(\gamma)(p)g \\ &= phg \\ &= pg(g^{-1}hg) \end{aligned}$$

So: the "holonomy at pg" is the holonomy at p <u>conjugated</u> by g. If we ignore the choice of p, then the holonomy gives not a specific $h \in G$ but a <u>conjugacy class</u> in G:

$$hol(\gamma) \in G/AD(G)$$

We can extract information about holonomies around loops using invariants of conjugacy classes. E.g. if

$$\rho : G \longrightarrow \mathrm{Aut}(V)$$

is a finite dimensional representation of G, we can define a trace on G:

$$tr_\rho(g) = tr(\rho(g)),$$

which is invariant under conjugation, by the cyclic property of the trace. This lets us define, for each $\gamma : [0,1] \longrightarrow M$,

$$W_\rho(\gamma) = tr_\rho(\mathrm{hol}(\gamma))$$

which is called a <u>Wilson loop</u>.

We'll see that under any gauge transformation of P, the holonomy h is also only changed by conjugation, so in fact the Wilson loops $W_\rho(\gamma)$ give <u>gauge invariant</u> information about the connection. The basic idea behind the "loop representation" of a gauge theory is to take Wilson loops as the essential variables.