D. Wise

Recall that a <u>connection</u> on a principal G-bundle $\begin{matrix} P \\ \downarrow \pi \\ M \end{matrix}$ is a smoothly varying family of linear maps (one for each $p \in P$):

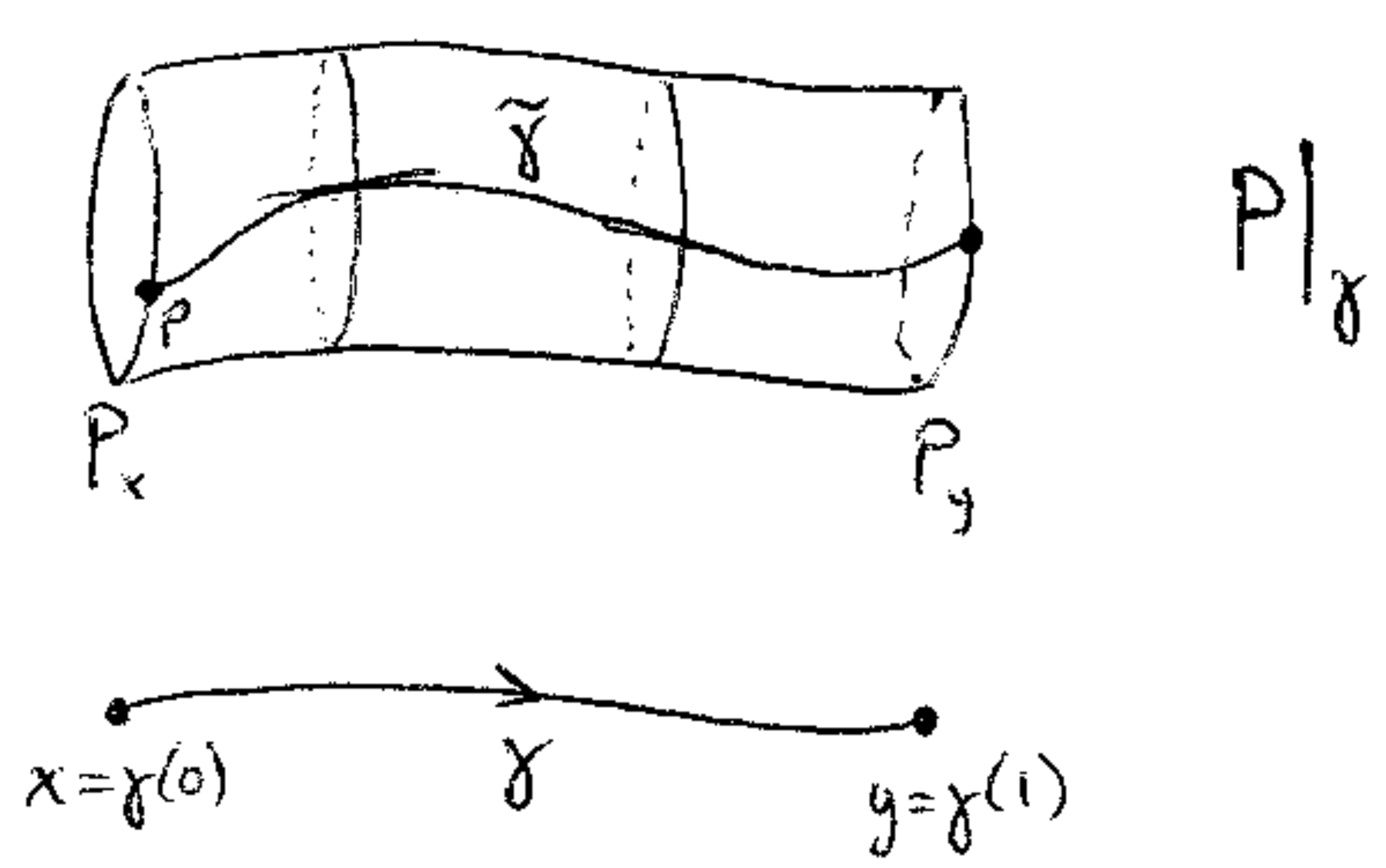
$$A_p : T_{\pi(p)} M \longrightarrow T_p P$$

s.t.

1) $\quad d\pi \circ A_p = 1_{T_{\pi(p)} M}$

2) $\quad A_{pg} = dR_g \circ A_p$

We saw that doing parallel translation along $\gamma : [0,1] \longrightarrow M$:



meant solving the ODE

$$\tilde{\gamma}'(t) = A_{\tilde{\gamma}(t)}(\gamma'(t)) \qquad (*)$$

for $\tilde{\gamma} : [0,1] \longrightarrow P$. We defined the <u>holonomy</u> of γ to be the isomorphism

$$\begin{aligned} \mathrm{hol}(\gamma) : P_{\gamma(0)} &\longrightarrow P_{\gamma(1)} \\ p &\longmapsto \tilde{\gamma}_p(1) \end{aligned}$$

where $\tilde{\gamma}_p$ is the unique solution of $(*)$ satisfying the initial condition $\tilde{\gamma}_p(0) = p \in P_{\gamma(0)}$.

Using parallel translation in a principal G-bundle P, we immediately get:

1) Parallel translation in any associated bundle

$$E = P \times_G V = \frac{P \times V}{(pg, v) \sim (p, gv)} :$$

If $\tilde{\gamma}_p(t)$ is the horizontal lift of γ starting at $p \in P_x$, then parallel translation of the point $[p, v] \in E_x$ along γ is given by

$$[\tilde{\gamma}_p(t), v] \in E_{\gamma(t)} .$$

Note this is well defined, since parallel translation of $[pg, g^{-1}v]$ $(=[p,v])$ along γ is given by

$$[\tilde{\gamma}_{pg}(t), g^{-1}v] = [\tilde{\gamma}_p(t)g, g^{-1}v] = [\tilde{\gamma}_p(t), v]$$

since we showed last time that horizontal lifts are related by $\tilde{\gamma}_{pg}(t) = \tilde{\gamma}_p(t)g$.

2) Covariant derivatives of sections of any associated vector bundle:

If $E = P \times_G V$ is a vector bundle (i.e. V is a vector space on which G acts linearly), then given $w \in T_x M$, choose γ s.t. $\gamma(0) = x$, $\gamma'(0) = w$.

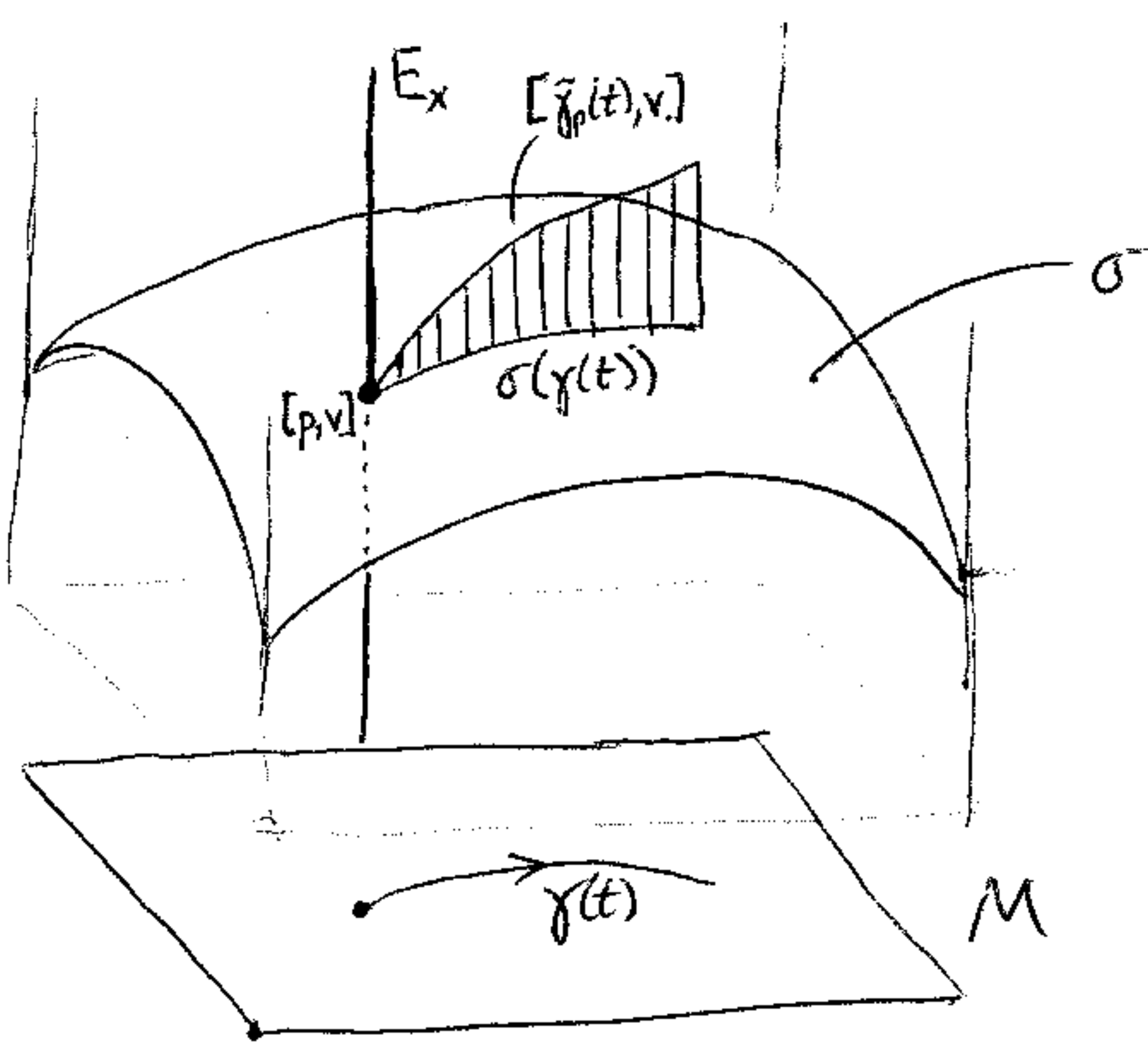
For a section $\sigma : M \to E$, define the covariant derivative

$$(D_w \sigma)(x) = \lim_{t \to 0} \frac{\sigma(\gamma(t)) - [\tilde{\gamma}_p(t), v]}{t} \in E_x$$

of σ at x in direction w.

For w a vector field, $(D_{w(x)}\sigma)(x)$ gives a new section:

$$E \downarrow \circlearrowleft D_w\sigma \quad M$$, the covariant derivative of σ w.r.t. w.



To make these ideas more concrete, let's see what connections and parallel translation amount to in a local trivialization. For notational simplicity, we just assume P is trivial:

$$\begin{array}{ccc} P = M \times G & & (x,g) \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow \\ M & & x \end{array}$$

By definition, a connection A gives a family of maps

$$\begin{array}{ccc} A_{(x,g)} : T_{\pi(x,g)} M & \longrightarrow & T_{(x,g)}(M \times G) \\ \| & & \| \\ T_x M & & T_x M \times T_g G \end{array}$$

satisfying:

1) $d\pi \circ A_{(x,g)} = 1_{T_x M}$.

Since $d\pi : T_x M \times T_g G \longrightarrow T_x M$ is given by $(v, X) \longmapsto v$, this equation just says $A_{(x,g)}(v) = (v, \tilde{A}_{(x,g)}(v))$ for some linear map

$$\tilde{A}_{(x,g)} : T_x M \longrightarrow T_g G$$

2) $A_{(x,g)g'} = dR_{g'} \circ A_{(x,g)}$

This is equivalent (Exercise!) to $\tilde{A}_{(x,g)} = dR_g \circ \tilde{A}_{(x,1)}$ where R_g here denotes the right action of G on itself. I.e.

$$\begin{tikzcd} T_x M \arrow[r, "\tilde{A}_{(x,1)}"] \arrow[rr, bend right, "\tilde{A}_{(x,g)}"'] & T_1 G \arrow[r, "dR_g"] & T_g G \end{tikzcd}$$ commutes.

So, all of the $\tilde{A}_{(x,g)}$ are determined by

$$A_x := \tilde{A}_{(x,1)} : T_x M \longrightarrow \mathfrak{g}$$

where $\mathfrak{g} = T_1 G$ is the Lie algebra of G.

So, on a trivial principal G-bundle (or in a local trivialization), a connection amounts to a smooth map

$$A : TM \longrightarrow \mathfrak{g}$$

that is linear on each T_xM. In other words, it's a Lie algebra-valued 1-form on M.

<u>Exercise</u>: Show that on the trivial principal G-bundle P, the ODE for parallel translation reduces to

$$\tilde{\gamma}_G'(t) = dR_{\tilde{\gamma}_G(t)} \circ A_{\gamma(t)}(\gamma'(t))$$

where the horizontal lift of γ to $P = M \times G$ is given by

$$\begin{aligned} \tilde{\gamma} : [0,1] &\longrightarrow P \\ t &\longmapsto (\gamma(t), \gamma_G(t)) \end{aligned}$$

and we take $\gamma_G : [0,1] \longrightarrow G$ s.t. $\gamma_G(0) = 1 \in G$.

(Note that $dR_{\tilde{\gamma}_G(t)} \circ A_{\gamma(t)} : T_{\gamma(t)}M \longrightarrow \mathfrak{g} \longrightarrow T_{\tilde{\gamma}_G(t)}G$

and the claim is: $\underset{\substack{\text{velocity}\\\text{of path}\\\text{on } M}}{\gamma'} \longmapsto \underset{\substack{\text{velocity of}\\\text{path in } G.}}{\tilde{\gamma}_G'}$)

Next, if $P = M \times G$, then any associated bundle can be trivialized in a canonical way:

$$\begin{aligned} E = P \times_G V &\xrightarrow{\sim} M \times V \\ [(x,g), v] &\longmapsto (x, gv) \end{aligned}$$

(Exercise: check that this is a bundle isomorphism). Under this identification, parallel translation in E is given by

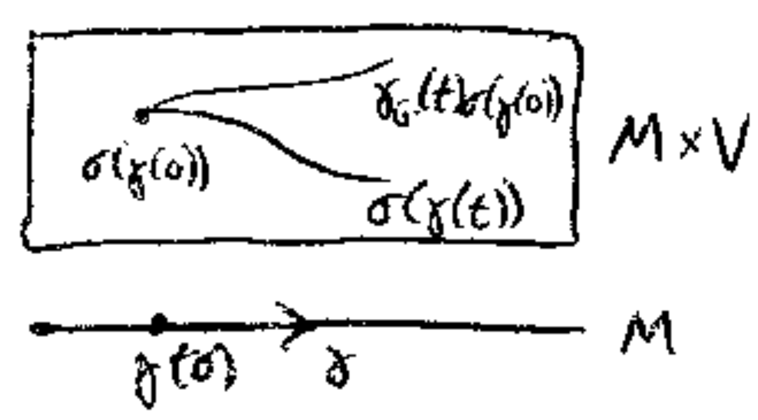
$$[(\gamma(t), \tilde{\gamma}_G(t)), v] \quad = \quad (\gamma(t), \tilde{\gamma}_G(t) v) \in M \times V.$$

I.e. parallel translation of $v \in V$ along γ is given by

$$\begin{aligned} [0,1] &\longrightarrow V \\ t &\longmapsto \tilde{\gamma}_G(t) v \end{aligned}$$

where $\tilde{\gamma}_G(t) \in G$ acts on $v \in V$ via the action that defines the associated bundle.

If E is a <u>vector</u> bundle, and σ is a section, we can use the trivialization $E = M \times V$ to think of σ as a function $\sigma : M \longrightarrow V$, and define



$$\begin{aligned} D_{\gamma'(0)}\sigma &:= \lim_{t \to 0} \frac{\sigma(\gamma(t)) - \tilde{\gamma}_G(t)\sigma(\gamma(0))}{t} \\ &= \lim_{t \to 0} \left(\frac{\sigma(\gamma(t)) - \sigma(\gamma(0))}{t} - \frac{(\tilde{\gamma}_G(t) - 1)\sigma(\gamma(0))}{t} \right) \\ &= \underbrace{\gamma'(0)[\sigma]}_{\substack{\text{directional} \\ \text{derivative of} \\ \sigma : M \to V}} - \tilde{\gamma}'_G(0)\, \sigma(\gamma(0)) \end{aligned}$$

note $\tilde{\gamma}_G(0) = 1$

elt. of $\mathfrak{g}$ elt. of V

action of $\mathfrak{g}$ on V is derivative of action of G on V defining the associated bundle E.

We can simplify this further, since we know $\tilde{\gamma}_G$ satisfies the IVP:

$$\tilde{\gamma}_G'(t) = dR_{\tilde{\gamma}_G(t)} \circ A_{\gamma(t)}(\gamma'(t))$$
$$\tilde{\gamma}_G(0) = 1 \in G.$$

Using this, we get

$$D_{\gamma'(0)}\sigma = \gamma'(0)\sigma - A_{\gamma(0)}(\gamma'(0)).$$

Working in local coordinates x^μ, if we take $\gamma'(0) = \frac{\partial}{\partial x^\mu}$ and define $D_\mu := D_{\partial_\mu}$ and $A_\mu(x) = A_x(\partial_\mu)$, we get

$$D_\mu \sigma = \partial_\mu \sigma - A_\mu \sigma.$$

as the formula for the covariant derivative in local coordinates. (Note: sometimes people call A "$-A$" instead, to get a "+" sign in the above formula — this is just a convention.)