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# BF THEORY & 3D GRAVITY

BF Theory — which has 3D gravity as a special case — is one of the simplest kinds of gauge theories. Its essential ingredients are:

1) A smooth n-dim manifold $M$

- not equipped with any metric — BF theory is a "background free" gauge theory, meaning there's no fixed background geometry on $M$
- typically, $M$ is diffeomorphic to $\mathbb{R} \times \Sigma$ where $\Sigma$ is an $(n-1)$-dim mfld, "space".

2) The gauge group $G$ — some Lie group, with Lie algebra $\mathfrak{g}$. We need $\mathfrak{g}$ to have a symmetric bilinear form

$$k : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$$

which is nondegenerate:

$$k(X,Y) = 0 \quad \forall Y \implies X = 0,$$

and invariant under the adjoint action

$$\begin{aligned} \mathrm{Ad} : G \times \mathfrak{g} &\longrightarrow \mathfrak{g} \\ (g, X) &\longmapsto gXg^{-1} \end{aligned}$$

i.e.:

$$k(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) = k(X,Y) \qquad \forall g \in G$$

Typically $G$ is semi-simple, and $k$ is proportional to the Killing form.

Some interesting examples:

- $G = U(1)$ — BF theory is like "background free electromagnetism"
- $n=3$, $G = SO(3)$ or $SU(2)$ — BF theory is 3d Riemannian gravity
- $n=3$, $G = SO(2,1)$ or $SL(2,\mathbb{R})$ — BF theory is 3d (Lorentzian) gravity
- $n=4$, $G = SO(3,1)$ or $SL(2,\mathbb{C})$ — BF theory is 4d "topological gravity."

3) A connection $A$ on a principal $G$-bundle $\begin{array}{c} P \\ \downarrow \pi \\ M \end{array}$.

<u>Locally</u>, we've seen that $A$ can be viewed as a Lie algebra-valued 1-form, i.e. a smooth map

$$A : TM \longrightarrow \mathfrak{g}$$

whose restriction to any tangent space $T_x M$ is linear.

4) The <u>curvature</u> $F$ of $A$. Let's describe this in local coordinates, where $F$ can be viewed as a Lie algebra-valued 2-form. There are two ways of using $A$ to get $\mathfrak{g}$-valued 2-forms:

$$dA : \Lambda^2 TM \longrightarrow \mathfrak{g}$$ — take the differential of the form part of $A$

$$[A,A] : \Lambda^2 TM \longrightarrow \mathfrak{g}$$ — wedge the form parts, take Lie bracket of $\mathfrak{g}$ parts

Explicitly, choosing a basis $\{\xi_I\}$ for $\mathfrak{g}$, we can write

$$A = A^I \xi_I$$

where each $A^I : TM \longrightarrow \mathbb{R}$ is an ordinary 1-form, and then

$$dA = (dA^I)\xi_I$$

$$[A,A] = (A^I \wedge A^J)[\xi_I, \xi_J]$$
$$= C^K_{IJ}(A^I \wedge A^J)\xi_K$$

where $C^K_{IJ}$ are the structure constants for $\mathfrak{g}$.

Locally, the curvature $F$ is then the $\mathfrak{g}$-valued 2-form

$$F = dA + \tfrac{1}{2}[A,A]$$

Globally, $F$ is a 2-form with values in the vector bundle

$$\mathrm{Ad}(P) := P \times_{\mathrm{Ad}} \mathfrak{g}$$

i.e. a vector bundle map

$$\begin{tikzcd} \Lambda^2 TM \arrow[rr, "F"] \arrow[dr] & & \mathrm{Ad}(P) \arrow[dl] \\ & M & \end{tikzcd}$$

5) An $\mathrm{Ad}(P)$-valued $(n-2)$-form $E$. Locally, $E$ is just a $\mathfrak{g}$-valued $(n-2)$-form.

6) The action:

$$S[A,E] = \int_M k(E,F)$$

Here $k(E,F)$ means wedge the form parts and apply $k$ to $\mathfrak{g}$-parts:

$$\begin{aligned} k(E,F) &= k(E^I \xi_I, F^J \xi_J) \\ &= k(\xi_I, \xi_J)\, E^I \wedge F^J \end{aligned}$$

So $k(E,F)$ is an ordinary $n$-form — exactly the kind of thing we know how to integrate over $M$.

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Let's derive the equations of motion:

$$\delta S = \int_M k(\delta E, F) + k(E, \delta F)$$

For this to vanish for all $\delta E$, since $k$ is non-degenerate, we must have:

$$\boxed{F = 0}$$

The other EOM takes a bit more work. First:

$$\begin{aligned} \delta F &= \delta\left(dA + \tfrac{1}{2}[A,A]\right) \\ &= d\delta A + [A, \delta A] \\ &= d_A \delta A \end{aligned}$$

where $d_A = d + [A,\ ]$ is the <u>exterior covariant derivative</u>.

Exercise: Use the invariance of $k$ under Ad, ~~and~~ integration by parts to show that, up to a boundary term,

$$\int_M k(E, d_A \delta A) = \int_M k(d_A E, \delta A)$$

Conclude that the second EOM for BF theory is

$$\boxed{d_A E = 0}$$

So: the equations of BF theory say:

- $A$ is flat (i.e. its curvature is zero), &
- $E$ is covariantly closed.

Before studying these equations further, it will be good to have an example in mind:

3D GRAVITY: This is a BF theory where:

- $n = 3$; often $M \cong \mathbb{R} \times \Sigma$ with $\Sigma$ a surface
- $G = SO(2,1)$, the 3d Lorentz group (or $SL(2,\mathbb{R})$, its double cover)
- $A$ is an $SO(2,1)$-connection
- $E$ is called the coframe field (or the "dreibein")

Let's talk about the role of the coframe field first. Locally, it's a Lie algebra-valued 1-form (since $n=3$):

$$E : TM \longrightarrow \mathfrak{so}(2,1).$$

Here

$$\mathfrak{so}(2,1) = \left\{ \begin{pmatrix} 0 & p_2 & p_1 \\ p_2 & 0 & p_0 \\ p_1 & -p_0 & 0 \end{pmatrix} \ : \ p_0, p_1, p_2 \in \mathbb{R} \right\}$$

This is a 3D vector space, but better yet, it has an inner product

$$k(X, X') = \tfrac{1}{2} \operatorname{tr}(XX')$$

that has Lorentzian signature:

$$X = \begin{pmatrix} 0 & p_2 & p_1 \\ p_2 & 0 & p_0 \\ p_1 & -p_0 & 0 \end{pmatrix} \implies k(X,X) = -p_0^2 + p_1^2 + p_2^2$$

and is invariant under the adjoint action of $SO(2,1)$. This means $\mathfrak{so}(2,1) \cong \mathbb{R}^{2,1}$, 3D Minkowski space, and $SO(2,1)$ acts as Lorentz transformations. So, we can think of $E$ as

$$E : TM \longrightarrow \mathbb{R}^{2,1}.$$

This explains the name "coframe field": A <u>frame</u> for us was an isomorphism like $\mathbb{R}^{2,1} \longrightarrow T_xM$, whereas, at each $x$, $E$ provides a <u>coframe</u> $T_xM \longrightarrow \mathbb{R}^{2,1}$.

Gravity is often written in terms of a metric on $M$, rather than a coframe field. (Recall that a <u>metric</u> is a smoothly varying inner product — here of Lorentzian signature — on the tangent spaces of $M$.) In the BF version of 3D gravity, the metric is simply the pullback of $k$ to each $T_x M$ along $E$. That is:

$$g(v,w) = k(E(v), E(w)) \qquad v, w \in T_x M.$$

Next time we'll look closer at the equations

$$F = 0$$
$$d_A E = 0$$

and their geometric meaning, esp. for 3D gravity.