GAUGE THEORY & TOPOLOGY

(Spring 2005)

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29 March 2005

Topological Gauge Theory

We want to compare "topological" gauge theories, ones that don't involve a metric on spacetime:

EF theory $\qquad S = \int tr(E \wedge F)$

(e.g. gravity in 3d spacetime)

General Relativity $\qquad S = \int tr(\underbrace{e \wedge \cdots \wedge e}_{\substack{n-2 \text{ of these} \\ \text{in } n\text{-dim spacetime}}} \wedge F)$

to gauge theories that do involve a metric on spacetime:

Yang-Mills Theory
(e.g. Maxwell's Eqns) $\qquad S = \int tr(F \wedge {*F})$

(Hodge star operator
— involves a metric
on spacetime.

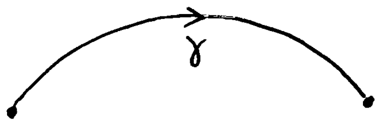
— more precisely, a metric treated as nondynamical "background structure."

To do this comparison, we need to generalize what we've done in fall & winter, replacing discrete structures by continuum structures:

triangulated manifold (spacetime) ⟶ smooth manifold

finite group (gauge group) ⟶ Lie group

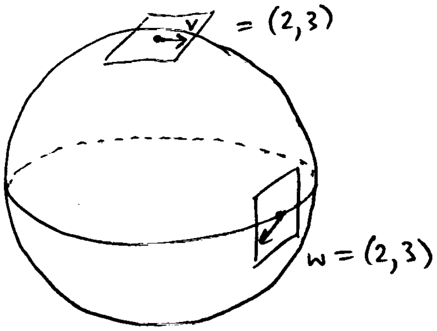
In particular, we need to introduce the concept of a "principal G-bundle" for a Lie group. Previously, we said that a connection on a (triangulated) manifold M assigns to each (edge) path γ a group element $A(\gamma) \in G$ describing how a particle's state transforms as we "parallel transport" it along γ.



$A(\gamma) \in G$

The problem is, it doesn't really make sense to assert that the state of a particle at x is "equal" to the state of a particle at y (if $x \neq y$). So saying "$A(\gamma) = 1 \in G$" makes no sense if $x \neq y$, unless we choose some extra structure to help us make this decision.

To do things more correctly, we need the concept of principal G-bundle — most easily found by considering an example involving tangent vectors and "frames" on a smooth manifold. We can think of a tangent vector $v \in T_xM$ to a point x in a smooth manifold M as a list of n real numbers, & use these lists to say whether $v \in T_xM$ is "equal" to $w \in T_yM$, <u>if</u> you pick bases for T_xM & T_yM:



But such statements <u>depend</u> on the arbitrary choice of bases! So, let's admit this and formalize it, by saying:

<u>Def</u> — A <u>frame</u> for an n-dimensional vector space V is an ordered basis $e_1, \dots, e_n$ of V.

<u>Def</u> — A <u>frame at x</u> where $x \in M$ (M a smooth manifold) is a frame for T_xM.

So we can say $v \in T_xM$ & $w \in T_yM$ are "equal" only after having chosen frames at x & y.

We can think of a frame for V in two other equivalent ways:

$$f : V \xrightarrow{\sim} \mathbb{R}^n \qquad \text{(a \underline{coframe})}$$

(a linear isomorphism) given by

$$f : a_1e_1 + \cdots + a_ne_n \longmapsto (a_1, \dots, a_n)$$

or

$$e = f^{-1} : \mathbb{R}^n \xrightarrow{\sim} V \qquad \text{(a \underline{frame})}$$

given by

$$e : (a_1, \dots, a_n) \longmapsto a_1 e_1 + \cdots + a_n e_n$$

In these terms, we say $v \in T_xM$ & $w \in T_yM$ are the "same" if $f(v) = f'(w)$:

$$\begin{tikzcd}
v \in T_xM \arrow[dr, "f", "\text{frame at } x"'] & & w \in T_yM \arrow[dl, "f'"', "\text{frame at } y"] \\
& \mathbb{R}^n &
\end{tikzcd}$$

or $w = f' f^{-1} v$.

Now:

<u>Def</u> – Let F_xM be the set of all frames at $x \in M$.

If $M = \mathbb{R}^n$, then we have a specific diffeomorphism

$$F_xM \cong GL(n)$$

where $GL(n) = \{n \times n \text{ matrices with } \det \neq 0\}$ is a Lie group. In general however, when $M \neq \mathbb{R}^n$, we don't have a God-given diffeomorphism $F_xM \cong GL(n)$, &

we don't have a God-given way of multiplying frames. BUT, we can "divide" them: given two frames $f, f' : T_x M \xrightarrow{\sim} \mathbb{R}^n$ we can compare them and get an element of $GL(n)$:

$$\begin{tikzcd}
T_x M \arrow[d, bend right=60, "f"', "\wr"] \arrow[d, bend left=60, "f'", "\wr"'] \\
\mathbb{R}^n
\end{tikzcd}$$

$f' f^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is an element of $GL(n)$. Or: use one frame to write the basis vectors in the other frame as lists of numbers, & get a matrix in $GL(n)$. If we formalize this notion, we get the concept of a "G-torsor" (here $G = GL(n)$), which is a set where we can divide one elt by another & get an elt of G.

There's something else going on too: if we have a frame $f : T_x M \to \mathbb{R}^n$ & an elt $g \in GL(n)$, we get a new frame gf as follows:

$$\begin{tikzcd}
T_x M \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & \mathbb{R}^n \arrow[r, "g"] & \mathbb{R}^n
\end{tikzcd}$$

So we can multiply frames & elts of $GL(n)$ to get frames!

Moreover:

$$\underbrace{(f'f^{-1})}_{\in GL(n)}f = f'$$

"division" and "multiplication" are nicely related as above.

So:

<u>Def</u> – Given a Lie group G, a <u>G-torsor</u> T is a manifold on which G acts:

$$\begin{aligned} G \times T &\longrightarrow T \\ (g,t) &\longmapsto gt \end{aligned}$$

(with $1t = t$, $g(ht) = (gh)t$) such that for any t, t' there's a unique element $t't^{-1} \in G$ with

$$(t't^{-1})t = t'$$

so we also have

$$\begin{aligned} T \times T &\longrightarrow G \\ (t',t) &\longmapsto t't^{-1} \end{aligned}$$

(and we want the maps $G \times T \to T$, $T \times T \to G$ to smooth). We also require that T be nonempty!

(Note: the last clause, while obnoxious, rules out what otherwise would be a G-torsor – the empty set!)

(This is a lousy definition of a torsor – we'll get to better ones later)

So: if M is an n-manifold & $x \in M$, $F_x M$ is a $GL(n)$-torsor. We then define

$$FM = \bigcup_{x \in M} F_x M$$

which becomes a smooth manifold equipped with a map

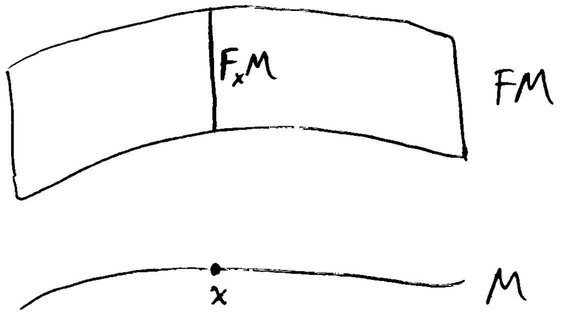
$$\begin{tikzcd} FM \arrow[d, "p"'] \\ M \end{tikzcd}$$

sending frames at x to x. This is called the <u>frame bundle</u> of M & motivates the definition of a "principal G-bundle." FM keeps track of all possible frames at all possible points $x \in M$, &

$$p^{-1}(x) = F_x M$$

is a $GL(n)$-torsor.

So it's a "bundle of $GL(n)$-torsors":



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We saw last time that for any n-dimensional vector space V there is a set of frames (i.e. ordered bases) for V:

$$FV = \{ f : V \xrightarrow{\sim} \mathbb{R}^n \}$$

& saw that this is a <u>torsor</u> for the group $GL(n)$. We gave a clunky definition of torsor last time; here are two nicer (but equivalent) ones:

<u>Def</u> - Given a (Lie) group G, a <u>G-torsor</u> is a set (manifold) on which G acts (smoothly), where the action is free & transitive

an action of G on T is free if $\forall t, t' \in T$, there is at most one $g \in G$ s.t. $gt = t'$;

an action of G on T is transitive if $\forall t, t' \in T$, there is at least one $g \in G$ s.t. $gt = t'$.

& T is nonempty.

Note that G itself is a G-torsor where the action of G on G is left multiplication: $\forall h, h' \in G$ $\exists !\, g \in G$ s.t. $gh = h'$ & G is nonempty since $1 \in G$. Also: any G-torsor T is isomorphic to G & we get an isomorphism by picking $t_0 \in T$ & then define

$$f : G \xrightarrow{\sim} T$$

by $$f(g) = g t_0 \in T$$

f is one-to-one because the action of G on T is free; f is onto because the action is transitive. Moreover f preserves the action of G:

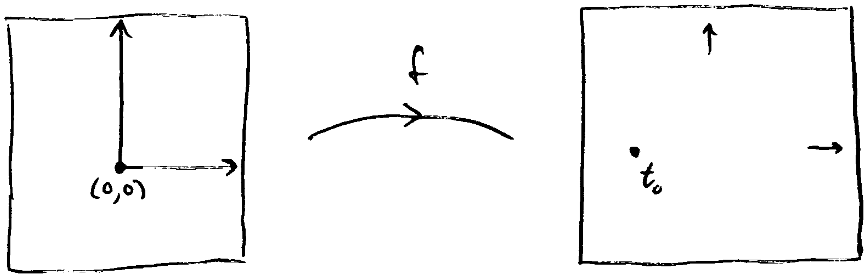
$$f(hg) = hf(g) \qquad \forall g, h \in G$$

since

$$f(hg) = (hg)t_0 = h(gt_0) = hf(t_0).$$

So we say f is an <u>isomorphism of G-torsors.</u>

E.g.



$\mathbb{R}^2$ an $\mathbb{R}^2$-torsor T

So: Every G-torsor is isomorphic to G, but not in a unique way. A G-torsor is like "G before we chose which element was the identity".

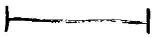
We're now ready for the even better definition:

<u>Def</u> — A <u>G-torsor</u> is a set T on which G acts which is isomorphic to G (as a G-set).

Note we now can drop the annoying "not empty" clause, since G is nonempty

Torsors are everywhere!

"Lengths" are not positive real numbers



but if we pick a unit of length ($t_0 \in T$) we can think of positive real numbers ($g \in G = \mathbb{R}^+$) as being lengths ($g t_0 \in T$). So lengths live in a G-torsor where $G = (\mathbb{R}^+, \cdot, 1)$. Similarly for all dimensionful quantities! For more on torsors:

http://math.ucr.edu/home/baez/torsors.html

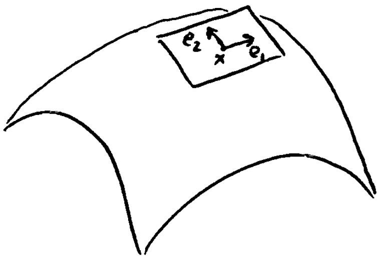
We saw that for any smooth n-dim manifold M, there is a <u>frame bundle</u>

$$FM = \bigcup_{x \in M} F_x M$$

where

$$F_x M = F(T_x M)$$

so a point in FM looks like



We have a map

$$\begin{tikzcd} FM \arrow[d, "p"'] \\ M \end{tikzcd}$$

such that $p^{-1}(x) = F_x M$ for each $x \in M$. Moreover, there's an action of $GL(n)$ on FM: a point in FM is a pair $x \in M$, $e : T_x M \xrightarrow{\sim} \mathbb{R}^n$ & given $g \in GL(n)$ it acts on (x,e) to give (x, ge).

$$\begin{tikzcd} T_x M \arrow[r, "e"] \arrow[rr, bend right, "ge"'] & \mathbb{R}^n \arrow[r, "g"] & \mathbb{R}^n \end{tikzcd}$$

In fact $GL(n)$ preserves each $F_x M \subseteq FM$ & these are $GL(n)$-torsors. So we say "FM is a bundle of $GL(n)$-torsors." More precisely,

<u>Def</u> – If M is a (smooth) manifold and G is a (Lie) group, a <u>bundle of G-torsors</u> over M is:

1) a (smooth) manifold P – the total space
2) a (smooth) map

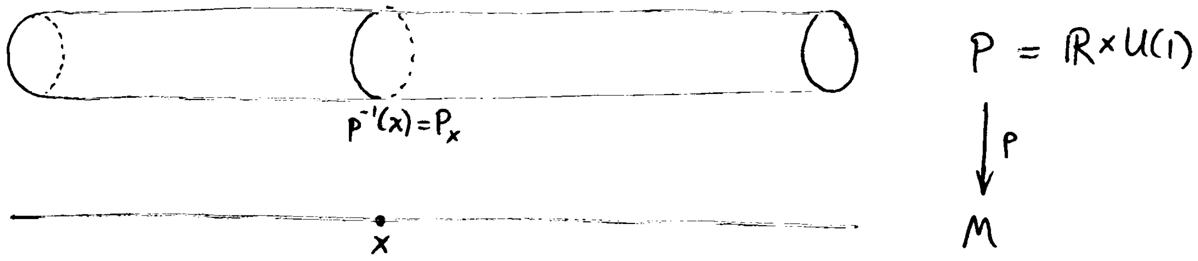
$$\begin{tikzcd} P \arrow[d, "p"] \\ M \end{tikzcd}$$ — the <u>projection</u>

3) a (smooth) action of G on P

such that the G-action preserves each <u>fiber</u> $p^{-1}(x)$, & each $p^{-1}(x)$ is a G-torsor (In fact we want $p^{-1}(x) \subseteq P$ to be a submanifold)

We often write the fiber $p^{-1}(x)$ as P_x. Note $P = \bigcup_{x \in M} P_x$.

Here's a picture of a $U(1)$-bundle over the manifold $\mathbb{R}$



This is a "trivial" bundle of G-torsors; we get such a thing by letting

$$P = M \times G$$

with

$$p(x,g) = x$$

& obvious action of G:

$$h(x,g) = (x, hg)$$

Warning: people usually define these concepts using a <u>right</u> action of G instead of left action. Our frame bundle would have a nice obvious right action of $GL(n)$ if we said

$$F_x M = \{ f : \mathbb{R}^n \xrightarrow{\sim} T_x M \}$$

instead of

$$F_x M = \{ f : T_x M \xrightarrow{\sim} \mathbb{R}^n \}$$

A "principal G-bundle" is a bundle of G-torsors that's "locally trivial":

Def – We say the bundle of G-torsors $\begin{array}{c}P\\ \downarrow{\scriptstyle p}\\ M\end{array}$ is <u>locally trivial</u> if $\forall x \in M$ $\exists$ open set $U \ni x$ s.t.

$$\begin{tikzcd}
p^{-1}U \subseteq P \arrow[rr, "t", "\sim"'] \arrow[dr, "p|_{p^{-1}U}"'] & & U \times G \arrow[dl, "\pi"] \\
& U &
\end{tikzcd}$$

there exists a diffeomorphism $t : p^{-1}U \xrightarrow{\sim} U \times G$ s.t. the diagram commutes & t preserves the action of G:

$$t(gp) = g\,t(p).$$

We call t a local trivialization.

<u>Def</u> – A principal G-bundle is a locally trivial bundle of G-torsors.

In fact, it's hard (impossible?) to think of bundles of G-torsors that <u>aren't</u> locally trivial. In particular, the frame bundle <u>is</u> locally trivial – so it's a principal $GL(n)$-bundle.