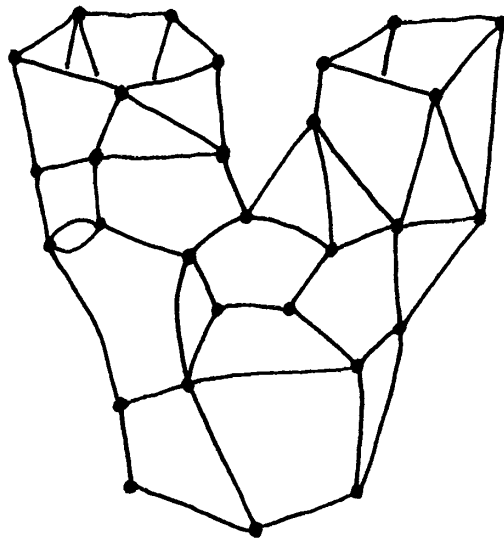


p -Form Electromagnetism on Discrete Spacetimes

Derek K. Wise

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derek@math.ucr.edu
<http://math.ucr.edu/~derek>

THE p -FORM GENERALIZATION OF ELECTROMAGNETISM

In electromagnetism, the basic physical field is a connection on a $U(1)$ bundle, locally a 1-form

$$A = A_\mu dx^\mu \quad \text{the 'gauge field'}$$

The A -field influences the motion of a charged particle:



$\int_\gamma A$ is a term in the action for the particle to move along the path γ .

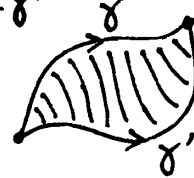
The connection is flat

$$\int_\gamma A = \int_{\gamma'} A \quad \text{for } \gamma \simeq \gamma'$$

\Updownarrow STOKES

$$F = dA = 0$$

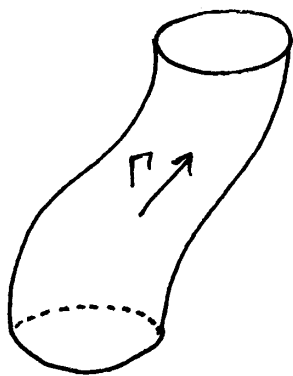
\curvearrowright curvature 2-form



In p -form electromagnetism, we generalize this story by promoting A to a p -form:

$$A = A_{\mu_1 \mu_2 \dots \mu_p} \frac{1}{p!} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_p}$$

This interacts naturally not with point particles, but with strings ($p=2$) or higher dimensional 'branes' ($p \geq 3$):



$\int_\Gamma A$ is a term in the action

The ' p -connection' is flat

\Updownarrow

$$\int A = 0 \quad \text{over any contractible } p\text{-sphere}$$

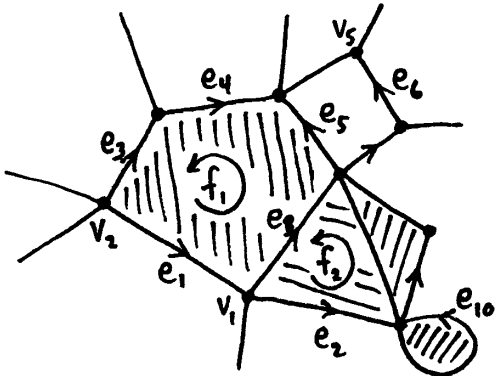
\Updownarrow STOKES

$$F = dA = 0$$

\curvearrowright curvature $(p+1)$ -form

DISCRETE SPACETIME AS A CHAIN COMPLEX

We can model discrete spacetime as some n -dimensional cell complex:



$X_0 = \{v_1, v_2, \dots\}$ = the set of vertices

$X_1 = \{e_1, e_2, \dots\}$ = the set of edges

$X_2 = \{f_1, f_2, \dots\}$ = the set of faces

\vdots
 X_n = the set of n -cells

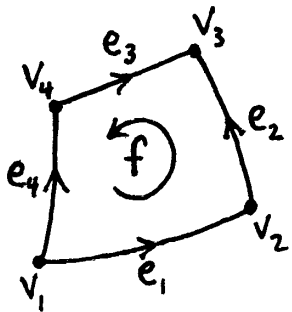
To describe discrete spacetime more algebraically, we let

$C_k :=$ the free abelian group on the set $X_k \cong \mathbb{Z}^{X_k}$

be the group of k -chains, and define boundary homomorphisms

$$\partial : C_k \longrightarrow C_{k-1}$$

in the obvious geometric way, e.g.:



has

$$\partial e_1 = v_2 - v_1 \in C_0$$

$$\partial f = e_1 + e_2 - e_3 - e_4 \in C_1$$

and hence

$$\partial \partial f = (v_2 - v_1) + (v_3 - v_2)$$

$$- (v_3 - v_4) - (v_4 - v_1) = 0 \in C_0$$

The principle that "the boundary of a boundary is zero" says we get a chain complex

$$0 \longleftarrow C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} \dots \xleftarrow{\partial} C_n$$

This is our model of discrete spacetime!

DISCRETE p -CONNECTIONS, CURVATURE, & GAUGE TRANSFORMATIONS

To get physical fields, we dualize our spacetime chain complex:

$$0 \leftarrow C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} \dots \xleftarrow{\partial} C_n$$

to get:
$$C^0 \xrightarrow{d} C^1 \xrightarrow{d} \dots \xrightarrow{d} C^n \rightarrow 0$$

where $C^k := \text{hom}(C_k, U(1))$ is the group of $U(1)$ -valued k -cochains and the coboundary maps d are defined by $df(c) := f(\partial c)$.

In lattice electromagnetism, the connection assigns to each edge an element of $U(1)$ — essentially the holonomy of the continuum connection. In the p -form generalization, we should get an elt. of $U(1)$ for each p -cell. We thus define a discrete $U(1)$ p -connection to be a homomorphism

$$A: C_p \rightarrow U(1)$$

so the group of p -connections on the chain complex C is

$$\mathcal{A}(C) := C^p.$$

Similarly, the field strength or curvature is the $(p+1)$ -cochain

$$F := dA: C_{p+1} \rightarrow U(1) \quad F \in C^{p+1}$$

Two p -connections are gauge equivalent if they differ by a coboundary:

$$A' \sim A \iff A' = A + d\varphi \quad \exists \varphi \in C^{p-1}$$

so we call

$$\mathcal{G}(C) := C^{p-1}$$

the group of gauge transformations.

EUCLIDEAN PATH INTEGRALS

An observable in discrete p -form electromagnetism is a gauge-invariant function of the connection:

$$f: \mathcal{A}(C)/g(C) \longrightarrow \mathbb{R}$$

A simple example is the real part of the curvature on some $(p+1)$ -cell c :

$$f(A) := \operatorname{Re} F(c) = \operatorname{Re} dA(c)$$

This is gauge invariant, since

$$f(A+d\varphi) = \operatorname{Re} d(A+d\varphi)(c) = \operatorname{Re} dA(c) = f(A).$$

In the quantum version of the theory, the observable f becomes a random variable with expected value given by the path integral

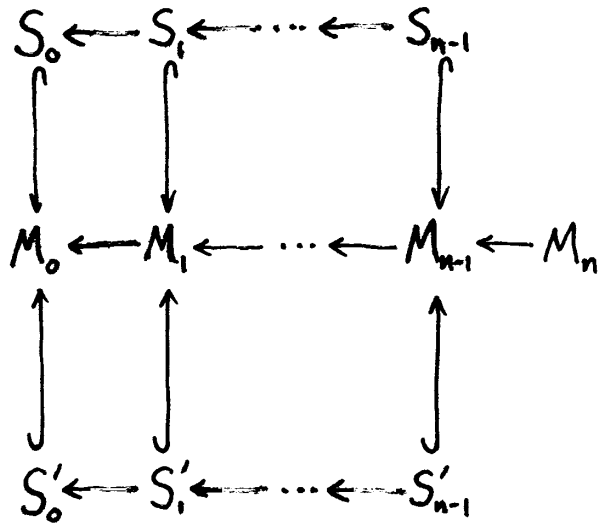
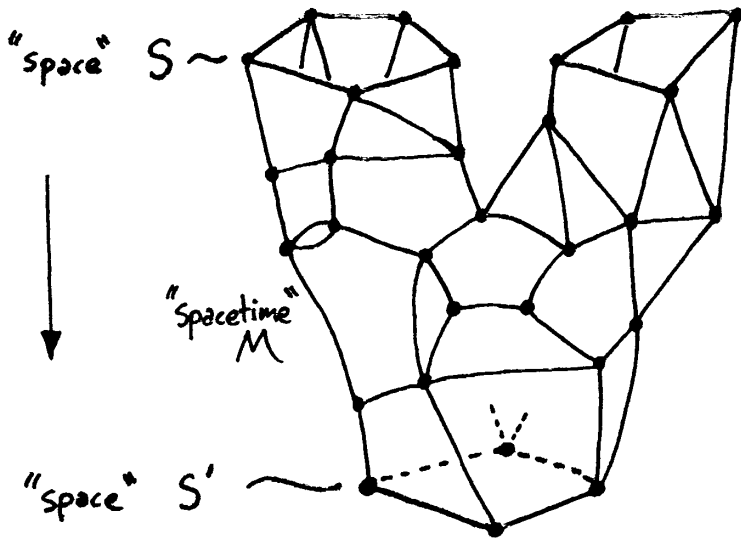
$$\langle f \rangle = \frac{\int_{\mathcal{A}(C)} f(A) e^{-S(A)} DA}{\int_{\mathcal{A}(C)} e^{-S(A)} DA}$$

where:

- " $e^{-S(A)}$ " $\in \mathbb{R}$ scales the relative probability for a particular connection $A: C_p \rightarrow U(1)$ to occur (see my paper [*] for details of how this is defined.)
- DA is a product of $U(1)$ -Haar measures, one for each p -cell.

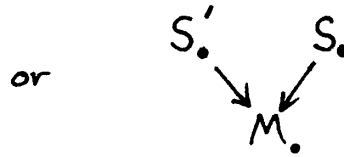
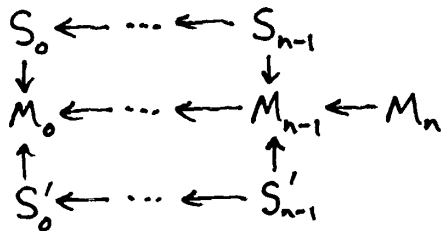
CHAIN COBORDISMS

To describe "time evolution" in discrete p-form electromagnetism, we'd like a notion of spacetime connecting slices of "space":



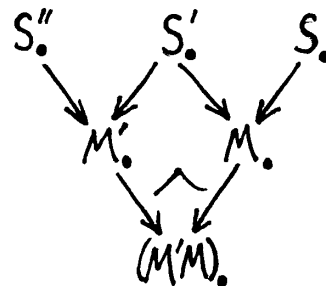
These form a nice category called $n\text{Chain}$:

- objects: $(n-1)$ -complexes $S_0 \leftarrow \dots \leftarrow S_{n-1}$
- morphisms: chain cobordisms



- composition: use pushouts

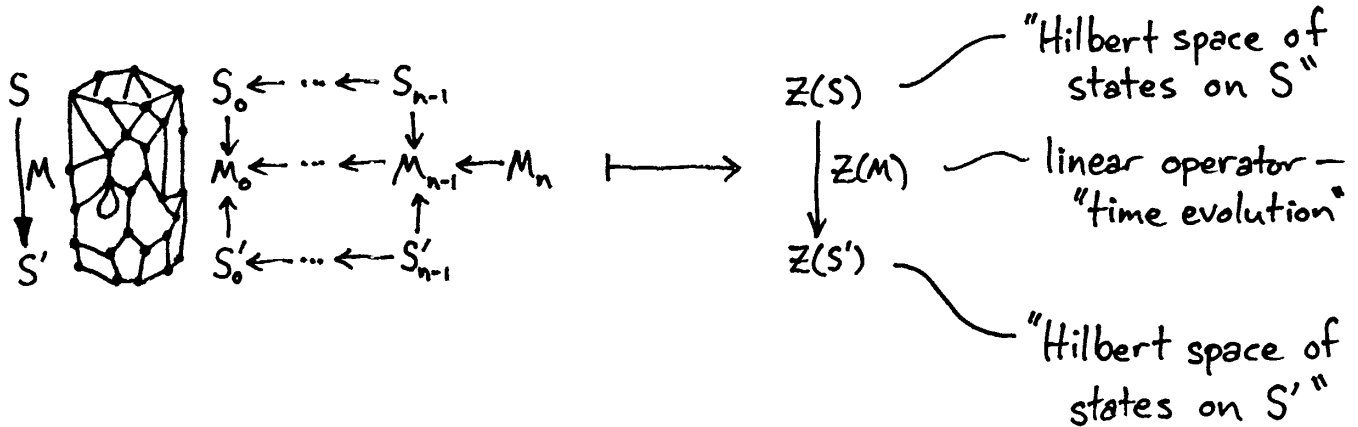
$$\left(\begin{array}{c} S''_0 \quad S'_0 \\ \swarrow \quad \searrow \\ M'_1 \end{array} \right) \circ \left(\begin{array}{c} S'_0 \quad S_0 \\ \swarrow \quad \searrow \\ M_1 \end{array} \right) =$$



CHAIN FIELD THEORY

In analogy to topological quantum field theory, I define a chain field theory to be a symmetric monoidal functor:

$$Z: n\text{Chain} \longrightarrow \text{Hilb}$$



Theorem: Discrete p -form electromagnetism is a chain field theory, with:

- for each object $S \in n\text{Chain}$,

$$Z(S) = L^2\left(\frac{A(S)}{g(S)}\right)$$

- for each chain cobordism $M: S \rightarrow S'$, the time evolution $Z(M): Z(S) \rightarrow Z(S')$ given by the path integral

$$\langle \psi, Z(M)\phi \rangle = \int_{A(M)} \bar{\psi}(A|_{S'}) \phi(A|_S) e^{-S(A)} DA .$$

(where DA is a product of $U(1)$ -Haar measures, suitably normalized.)

REFERENCES

This presentation is based on the paper:

[*] D. Wise, p-Form Electromagnetism on Discrete Spacetimes

available at:

<http://math.ucr.edu/~derek/pform>

Please see the bibliography of this paper for additional references.