Lattice p-Form Electromagnetism

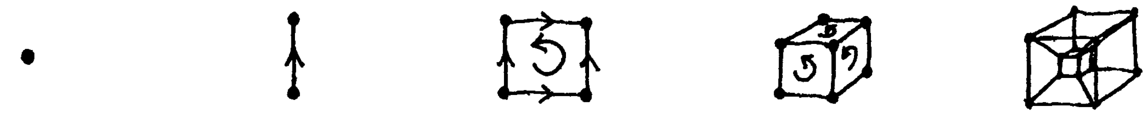
Derek K. Wise

1 September 2005

derek@math.ucr.edu
http://math.ucr.edu/~derek

Lattice Gauge Theory

- Approximate 4d Euclidean spacetime by a hypercubical lattice:

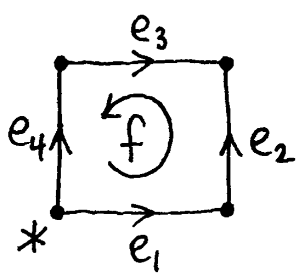


vertices	edges	square faces	cubes	hypercubes
– 'events'	– paths	– area elements	– volume elts.	– 4-vol. elts.

- Pick a gauge group G. The gauge field A is a 'discrete connection' – it assigns to each edge e an element

$$A(e) \in G$$

- To each face f, assign the curvature or field strength $F(f)$ using chosen orientation & base vertex $(*)$:



$$F(f) = A(e_1)A(e_2)A(e_3)^{-1}A(e_4)^{-1}$$

- Specify an action functional $S : [\text{connections } A] \longrightarrow \mathbb{R}$
  E.g. for $G = SU(2)$, Wilson used $S(A) = \sum_{\text{faces } f} [2 - \operatorname{tr} F(f)]$.

- Do path integrals: the expected value of an observable $\mathcal{O}$ is:

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{O}(A)\, e^{-S(A)}\, DA}{\int e^{-S(A)}\, \underbrace{DA}_{\text{Haar measure on } G^{[\text{Edges}]}}}.$$

p-Form Electromagnetism

Electromagnetism has abelian gauge gp. — usually $U(1)$ or $\mathbb{R}$ so the gauge field A is locally a 1-form

$$A = A_\mu dx^\mu$$

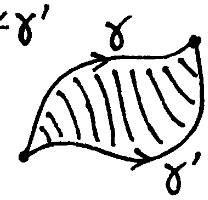
The A-field influences the motion of a charged particle:

$\int_\gamma A$ is a term in the <u>action</u> for the particle to move along the path γ.

The connection is flat

$$\Updownarrow$$

$$\int_\gamma A = \int_{\gamma'} A \quad \text{for } \gamma \simeq \gamma'$$



$$\Updownarrow \text{ STOKES}$$

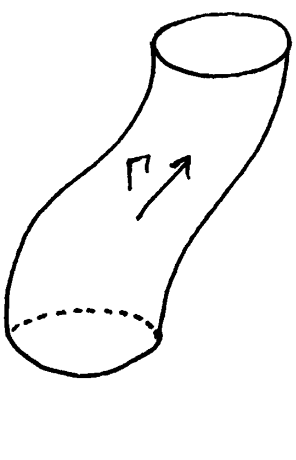
$$F = dA = 0$$

<u>curvature</u> 2-form

In p-form electromagnetism, we generalize this story by promoting A to a p-form:

$$A = A_{\mu_1 \mu_2 \cdots \mu_p} \frac{1}{p!} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p}.$$

This interacts naturally not with point particles, but with strings ($p=2$) or higher dimensional 'branes' ($p \geq 3$):



$\int_\Gamma A$ is a term in the action

The 'p-connection' is flat

$$\Updownarrow$$

$\int A = 0$ over any contractible p-sphere

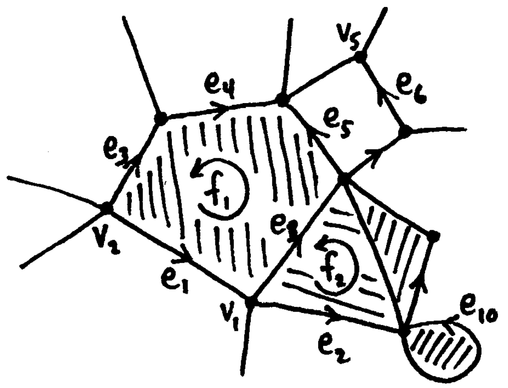
$$\Updownarrow \text{ STOKES}$$

$$F = dA = 0$$

<u>curvature</u> $(p+1)$-form

Discrete Spacetime as a Chain Complex

We can model discrete spacetime as some n-dimensional cell complex:



$X_0 = \{v_1, v_2, \ldots\}$ = the set of <u>vertices</u>
$X_1 = \{e_1, e_2, \ldots\}$ = the set of <u>edges</u>
$X_2 = \{f_1, f_2, \ldots\}$ = the set of <u>faces</u>
$\vdots$
X_n = the set of <u>n-cells</u>

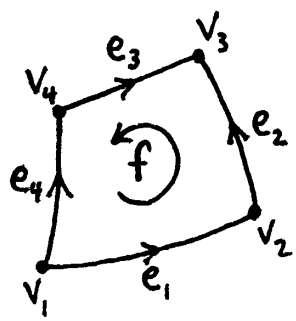
For G abelian we can describe discrete spacetime algebraically:

let $C_k :=$ the free abelian group on the set $X_k \cong \mathbb{Z}^{X_k}$

be the group of <u>k-chains</u>, and define <u>boundary homomorphisms</u>

$$\partial : C_k \longrightarrow C_{k-1}$$

in the obvious geometric way, e.g.:



has

$\partial e_1 = v_2 - v_1 \in C_0$
$\partial f = e_1 + e_2 - e_3 - e_4 \in C_1$
and hence
$\partial\partial f = (v_2 - v_1) + (v_3 - v_2) - (v_3 - v_4) - (v_4 - v_1) = 0 \in C_0$

The principle that "the boundary of a boundary is zero" says we get a <u>chain complex</u>

$$0 \longleftarrow C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} \cdots \xleftarrow{\partial} C_n$$

This is our model of discrete spacetime for abelian gauge theory!

Discrete p-Connections, Curvature, & Gauge Transformations

To get physical fields, we dualize our spacetime chain complex:

$$0 \longleftarrow C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} \cdots \xleftarrow{\partial} C_n$$

to get:

$$C^0 \xrightarrow{d} C^1 \xrightarrow{d} \cdots \xrightarrow{d} C^n \longrightarrow 0$$

where $C^k := \hom(C_k, U(1))$ is the group of $U(1)$-valued <u>k-cochains</u> and the <u>coboundary maps</u> d are defined by $df(c) := f(\partial c)$.

In lattice electromagnetism, the connection assigns to each <u>edge</u> an element of $U(1)$ — essentially the holonomy of the continuum connection. In the <u>p-form</u> generalization, we should get an elt. of $U(1)$ for each <u>p-cell</u>. We thus define a <u>discrete</u> <u>$U(1)$</u> <u>p-connection</u> to be a homomorphism

$$A : C_p \longrightarrow U(1)$$

so the group of p-connections on the chain complex C is

$$\mathcal{A}(C) := C^p.$$

Similarly, the <u>field strength</u> or <u>curvature</u> is the $(p+1)$-cochain

$$F := dA : C_{p+1} \longrightarrow U(1) \qquad F \in C^{p+1}$$

Two p-connections are <u>gauge</u> <u>equivalent</u> if they differ by a coboundary:

$$A' \sim A \iff A' = A + d\varphi \quad \exists \varphi \in C^{p-1}$$

so we call

$$\mathcal{G}(C) := C^{p-1}$$

the group of <u>gauge</u> <u>transformations</u>.

The Action

When $G=\mathbb{R}$, there is an obvious choice for the action

$$S:\hom(C_p,\mathbb{R})\to\mathbb{R}$$

Namely, since $C^{p+1}:=\hom(C_{p+1},\mathbb{R})\cong\mathbb{R}^{X_{p+1}}$ is a real vector space, we can give it an inner product $\langle -,-\rangle$ and let

$$S(A):=\langle dA,dA\rangle=\langle F,F\rangle$$

analogous to $\int F\wedge\star F$

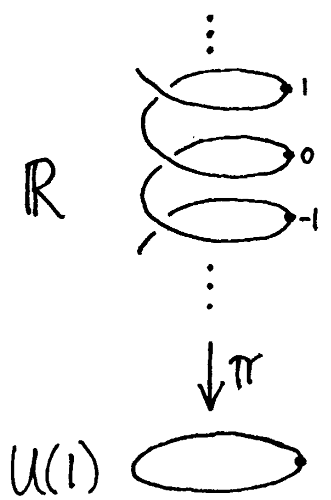
This leads to Gaussian path integrals:

$$Z=\int e^{-S}=\int e^{-\langle F,F\rangle}.$$

For $G=U(1)$, we could use the 'Wilson action':

$$S(A)=\sum_{c\in X_{p+1}}\left[1-\mathrm{Re}\,F(c)\right],$$

but this is hard to calculate with except numerically. Instead of defining S in the $U(1)$ theory, we define the analog of the Gaussian e^{-S} from the $\mathbb{R}$ theory. To do this we "wrap the real line around the circle":



This gives a short exact sequence

$$0\to\mathbb{Z}\hookrightarrow\mathbb{R}\xrightarrow{\pi}U(1)\to 0$$

An L^1 function $f:\mathbb{R}\to\mathbb{R}$ thus gives an L^1_{Haar} function $\hat{f}:U(1)\to\mathbb{R}$ by

$$\hat{f}(\theta)=\sum_{n\in\mathbb{Z}}f(\theta+2\pi n).$$

Gauge Groups for Lattice p-Form Electromagnetism

The short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow U(1) \longrightarrow 0$$

gives rise to a short exact sequence of chain maps

$$\begin{tikzcd}
 & & \vdots & \vdots & \vdots & \\
\text{curvatures} & 0 \arrow[r] & \hom(C_{p+1}, \mathbb{Z}) \arrow[u] \arrow[r] & \hom(C_{p+1}, \mathbb{R}) \arrow[u] \arrow[r] & \hom(C_{p+1}, U(1)) \arrow[u] \arrow[r] & 0 \\
\text{p-conns.} & 0 \arrow[r] & \hom(C_{p}, \mathbb{Z}) \arrow[u, "d"] \arrow[r] & \hom(C_{p}, \mathbb{R}) \arrow[u, "d"] \arrow[r] & \hom(C_{p}, U(1)) \arrow[u, "d"] \arrow[r] & 0 \\
\text{gauge trans.} & 0 \arrow[r] & \hom(C_{p-1}, \mathbb{Z}) \arrow[u, "d"] \arrow[r] & \hom(C_{p-1}, \mathbb{R}) \arrow[u, "d"] \arrow[r] & \hom(C_{p-1}, U(1)) \arrow[u, "d"] \arrow[r] & 0 \\
 & & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

"Lattice p-form e.m. with gauge gp. $\mathbb{Z}$"

"Lattice p-form e.m. with gauge gp. $\mathbb{R}$"

Path integral involves a Gaussian

$$e^{-\langle F,F\rangle}$$

"Lattice p-form e.m. with gauge gp. $U(1)$"

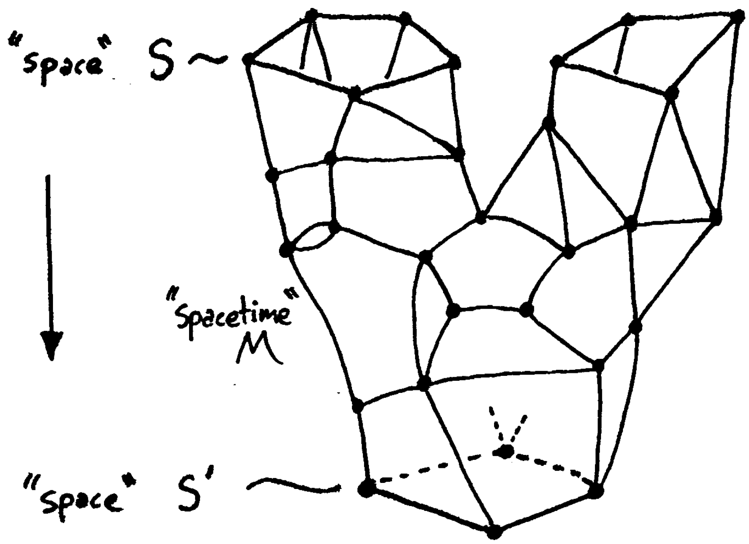
Path integral involves a 'wrapped Gaussian'

$$\sum_{n \in \mathbb{Z}^{X_{p+1}}} e^{-\langle F+2\pi n, F+2\pi n\rangle}$$

which is really a <u>theta function</u>!

Chain Cobordisms

To describe "time evolution" in discrete p-form electromagnetism, we'd like a notion of spacetime connecting slices of "space":



$$
\begin{tikzcd}
S_0 \arrow[d, hook] & S_1 \arrow[l] \arrow[d, hook] & \cdots \arrow[l] & S_{n-1} \arrow[l] \arrow[d, hook] & \\
M_0 & M_1 \arrow[l] & \cdots \arrow[l] & M_{n-1} \arrow[l] & M_n \arrow[l] \\
S'_0 \arrow[u, hook] & S'_1 \arrow[l] \arrow[u, hook] & \cdots \arrow[l] & S'_{n-1} \arrow[l] \arrow[u, hook] &
\end{tikzcd}
$$

These form a nice category called nChain:

- objects: $(n-1)$-complexes $S_0 \leftarrow \cdots \leftarrow S_{n-1}$
- morphisms: <u>chain cobordisms</u>

$$
\begin{tikzcd}
S_0 \arrow[d] & \cdots \arrow[l] & S_{n-1} \arrow[l] \arrow[d] & \\
M_0 & \cdots \arrow[l] & M_{n-1} \arrow[l] & M_n \arrow[l] \\
S'_0 \arrow[u] & \cdots \arrow[l] & S'_{n-1} \arrow[l] \arrow[u] &
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
S'_\bullet \arrow[dr] & & S_\bullet \arrow[dl] \\
& M_\bullet &
\end{tikzcd}
$$

- composition: use <u>pushouts</u>

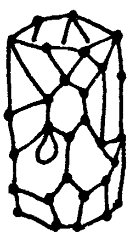
$$
\left(
\begin{tikzcd}
S''_\bullet \arrow[dr] & & S'_\bullet \arrow[dl] \\
& M'_\bullet &
\end{tikzcd}
\right) \circ \left(
\begin{tikzcd}
S'_\bullet \arrow[dr] & & S_\bullet \arrow[dl] \\
& M_\bullet &
\end{tikzcd}
\right) =
\begin{tikzcd}
S''_\bullet \arrow[dr] & & S'_\bullet \arrow[dl] \arrow[dr] & & S_\bullet \arrow[dl] \\
& M'_\bullet \arrow[dr] & & M_\bullet \arrow[dl] & \\
& & (M'M)_\bullet & &
\end{tikzcd}
$$

Chain Field Theory

In analogy to topological quantum field theory, I define a <u>chain field theory</u> to be a symmetric monoidal functor:

$$Z : n\mathrm{Chain} \longrightarrow \mathrm{Hilb}$$

$$\begin{array}{c} S \\ \big\downarrow M \\ S' \end{array}$$



$$\begin{tikzcd}
S_0 \arrow[d] & \cdots \arrow[l] & S_{n-1} \arrow[l] \arrow[d] & \\
M_0 & \cdots \arrow[l] & M_{n-1} \arrow[l] & M_n \arrow[l] \\
S'_0 \arrow[u] & \cdots \arrow[l] & S'_{n-1} \arrow[l] \arrow[u] &
\end{tikzcd}
\longmapsto
\begin{tikzcd}
Z(S) \arrow[d, "Z(M)"] \\
Z(S')
\end{tikzcd}$$

$Z(S)$ — "Hilbert space of states on S"

$Z(M)$ — linear operator — "time evolution"

$Z(S')$ — "Hilbert space of states on S'"

<u>Theorem</u>: Discrete p-form electromagnetism is a chain field theory, with:

- for each object $S \in n\mathrm{Chain}$,

$$Z(S) = L^2\left(\frac{\mathcal{A}(S)}{\mathcal{G}(S)}\right)$$

- for each chain cobordism $M : S \longrightarrow S'$, the time evolution $Z(M) : Z(S) \longrightarrow Z(S')$ given by the path integral

$$\langle \psi, Z(M)\phi \rangle = \int_{\mathcal{A}(M)} \overline{\psi}(A|_{S'})\,\phi(A|_S)\, e^{-S(A)}\, \mathcal{D}A \,.$$

(where $\mathcal{D}A$ is a product of $U(1)$-Haar measures, suitably normalized.)