Extended Topological Gauge Theory

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Work in progress with J. Morton

(Builds on Morton's earlier work:

math/0611930

0810.2361

+ 1 to appear shortly

and also influenced by previous work with:

A. Baratin	0910.1542
J. Baez, A. Baratin, L. Freidel	0812.4969
J. Baez, A. Crans	gr-qc/0603085

)

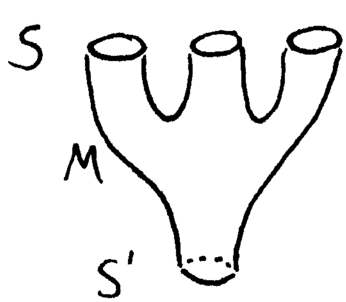
Extended Topological Field Theory

In *ordinary* topological quantum field theory (TQFT) (in the Atiyah/Segal axiomatic approach) we get:

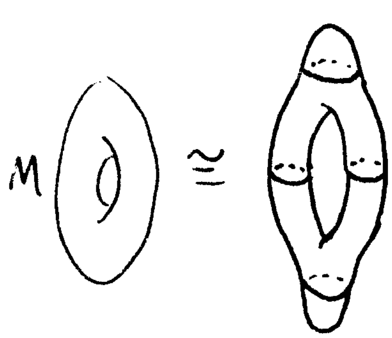
- Hilbert space for each $(n-1)$-manifold
- Linear map ("time evolution") for each n-dim cobordism (manifold w. boundary)

In a way that respects unions and gluing.

e.g. 2d TQFT



$$\begin{tikzcd} Z(S) \arrow[d, "Z(M)"] & \text{Hilb. Space} \arrow[l] \\ Z(S') & \end{tikzcd}$$

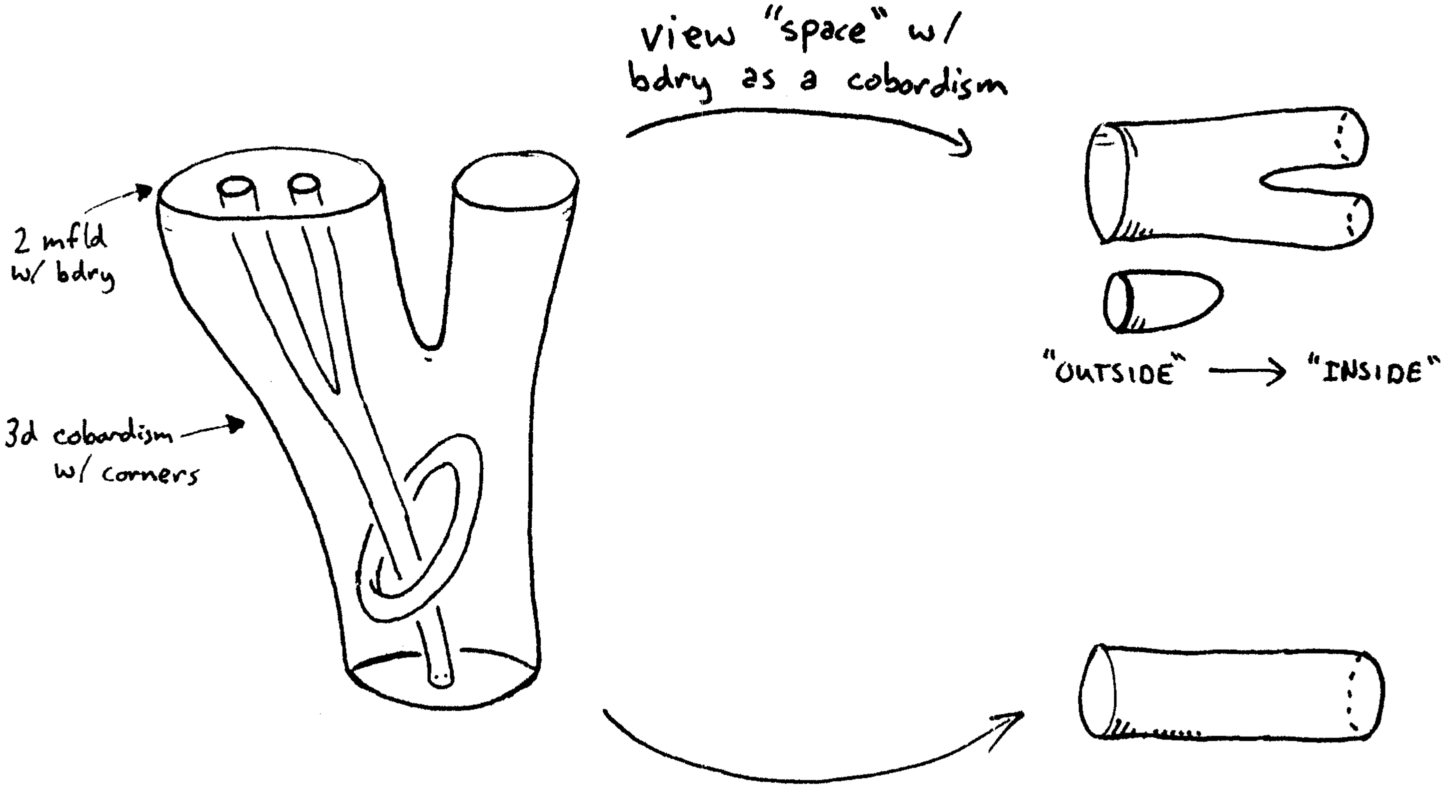


$$\begin{tikzcd} Z(\emptyset) = \mathbb{C} \arrow[d] \arrow[dddd, bend left, "Z(M)=\int e^{iS}"] \\ Z(S') \arrow[d] \\ Z(S' \cup S') \arrow[d] \\ Z(S') \arrow[d] \\ Z(\emptyset) = \mathbb{C} \end{tikzcd}$$

This gives us a "deeper look" at the partition function by breaking it into "stages."

In extended TQFT (eTQFT), the idea is to take this one step further, assigning some sort of algebraic data to:

- $(n-2)$-manifolds
- $(n-1)$-manifolds w. boundary (cobordisms)
- n-manifolds w. <u>corners</u> (cobordisms of cobordisms)



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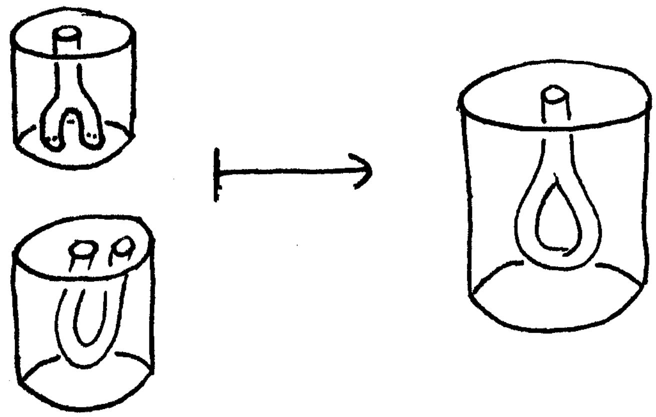
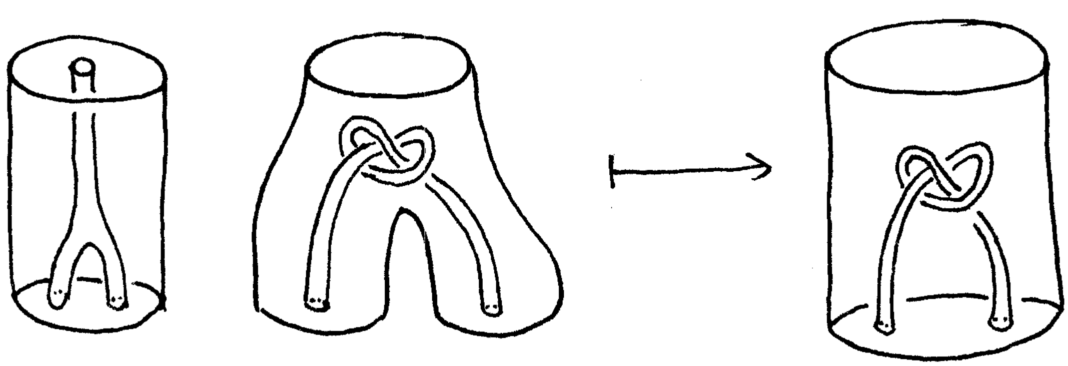
in a way that respects unions and two kinds of gluing operations:

"<u>vertical</u>":



"<u>horizontal</u>":



Criteria :

- When space has no boundary, eTQFT should reduce to TQFT (Hilbert spaces, linear maps)
  - Still want to do <u>physics</u>.

- Various consistency conditions that can be expressed elegantly using "higher-dimensional algebra."

Why?

- Topology applications: knot theory
- "Local" structure of TQFT:
  - Partition function ~ number-valued topological invariant <u>Z</u>
  - TQFT ~ can see <u>Z</u> as built up out of maps between <u>Hilbert spaces</u>
  - eTQFT ~ can see the <u>Hilbert spaces</u> as built up out of some sort of "local data"

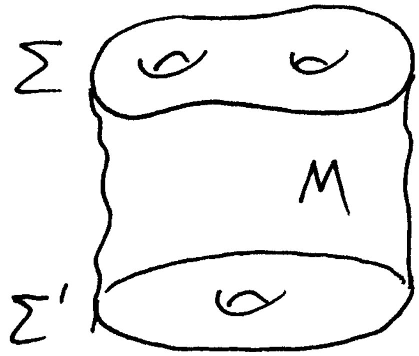
Typical Features of (un-extended) Topological Gauge Theory

Imagine quantizing a theory for a flat G-connection (e.g. BF theory)

Hilbert space of states on "space" Σ is typically

$$L^2(\underbrace{\mathcal{A}_0(\Sigma)/\mathcal{G}(\Sigma)}_{\text{moduli space of flat } G\text{-connections on } \Sigma})$$

and if we have a suitable action, we get a TQFT



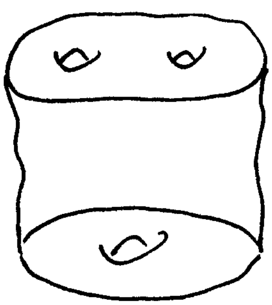
$$\begin{tikzcd}
Z(\Sigma) = L^2(\mathcal{A}_0(\Sigma)/\mathcal{G}(\Sigma)) \arrow[d, "Z(M)"] \\
Z(\Sigma') = L^2(\mathcal{A}_0(\Sigma')/\mathcal{G}(\Sigma'))
\end{tikzcd}$$

With $Z(M)$ calculated by the usual sort of path integral:

$$\langle \psi | Z(M) | \phi \rangle = \int_{\mathcal{A}_0(M)/\mathcal{G}(M)} \mathcal{D}A \; e^{iS[A]} \, \overline{\psi(A|_{\Sigma'})} \, \phi(A|_{\Sigma})$$

To understand eTQFT (à la J. Morton) it's helpful to think of this path integral in a different way:

A cobordism is really <u>two</u> maps (up to diffeo):



$$\begin{tikzcd} \Sigma \arrow[d] \\ M \\ \Sigma' \arrow[u] \end{tikzcd}$$

... which induces ...

$$\begin{tikzcd} \mathcal{A}_0(\Sigma)/\mathcal{G}(\Sigma) \\ \mathcal{A}_0(M)/\mathcal{G}(M) \arrow[u, "r_1"] \arrow[d, "r"'] \\ \mathcal{A}_0(\Sigma')/\mathcal{G}(\Sigma') \end{tikzcd}$$

"restriction"

... which in turn induces:

$$\begin{tikzcd} L^2(\mathcal{A}_0(\Sigma)/\mathcal{G}(\Sigma)) \arrow[d, "r_1^*"'] \\ \text{“}L^2_S(\mathcal{A}_0(M)/\mathcal{G}(M))\text{”} \\ L^2(\mathcal{A}_0(\Sigma')/\mathcal{G}(\Sigma')) \arrow[u, "r_2^*"] \end{tikzcd}$$

"pull back the wavefunction along the restriction map r_1."

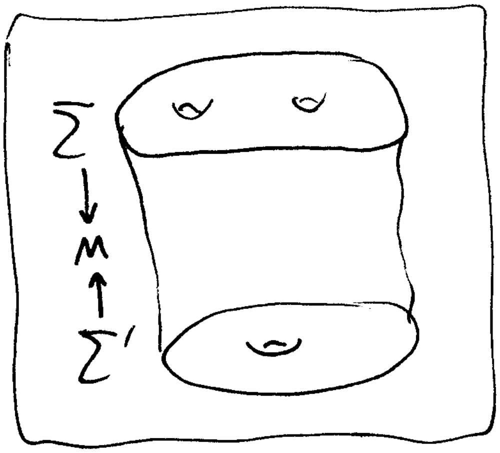
"Morally" a Hilbert space, but with "inner product"

$$(\xi|\zeta) = \int_{\mathcal{A}_0/\mathcal{G}} \mathcal{D}A\, e^{iS[A]}\, \overline{\xi(A)}\, \zeta(A)$$

Then: $$\langle\psi|Z(M)|\phi\rangle = \int \mathcal{D}A\, e^{iS[A]}\, \overline{\psi(A|_{\Sigma'})}\, \phi(A|_{\Sigma})$$
$$= (r_2^*\psi | r_1^*\phi)$$
$$= \langle\psi|(r_2^*)^\dagger r_1^*|\phi\rangle$$

So: $$Z(M) = (r_2^*)^\dagger r_1^* =: r_{2*} r_1^*$$

Summary :



$\xrightarrow{\text{TQFT}}$

$$\begin{tikzcd} Z(\Sigma) \in \mathrm{Hilb} \arrow[d, "Z(M) = r_{2*} r_1^*"] \\ Z(\Sigma') \end{tikzcd}$$

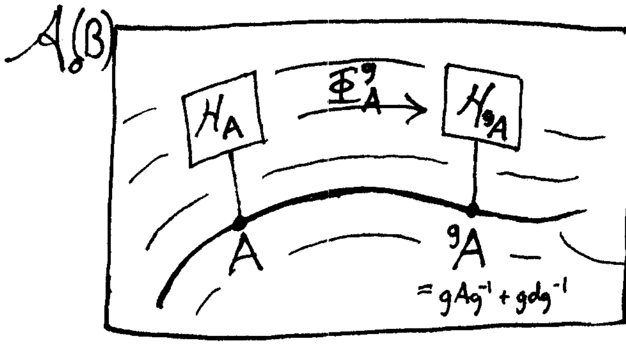
$$\begin{tikzcd} \mathcal{A}_0(\Sigma)/\mathcal{G}(\Sigma) \\ \mathcal{A}_0(M)/\mathcal{G}(M) \arrow[u, "r_1"'] \arrow[d, "r_2"] \\ \mathcal{A}_0(\Sigma')/\mathcal{G}(\Sigma') \end{tikzcd} \rightsquigarrow \begin{tikzcd} Z(\Sigma) = L^2(\mathcal{A}_0/\mathcal{G}) \arrow[d, "r_1^*"] \\ \text{“}L^2_s(\mathcal{A}_0/\mathcal{G})\text{”} \arrow[d, shift left, "r_{2*} = (r_2^*)^\dagger"] \\ Z(\Sigma') = L^2(\mathcal{A}_0/\mathcal{G}) \arrow[u, shift left, "r_2^*"] \end{tikzcd}$$

J. Morton has constructed certain <u>eTQFT's</u> essentially by mimicking this idea at the level of "space" ...

Morton's eTQFT from (flat) connections for (finite) gauge group G.

very roughly ...

- For an $(n-2)$-mfld B, instead of a Hilbert space, we get a "bundle" of Hilbert spaces over the space $\mathcal{A}_0(B)$ of flat connections, which is gauge equivariant



gauge orbit

$$\Phi^{g'g}_A = \Phi^{g'}_{{}^gA}\,\Phi^g_A$$

- For an $(n-1)$-mfld w. bdry, representing "space":

$$B_1 \longrightarrow \Sigma \longleftarrow B_2$$

we get a pair of induced maps

$$\begin{pmatrix}\text{Gauge-equivariant} \\ \text{bundles over } \mathcal{A}_0(B_1)\end{pmatrix} \xrightarrow{r_1^*} \begin{pmatrix}\text{bundles over} \\ \mathcal{A}_0(\Sigma)\end{pmatrix} \underset{r_{2*}}{\overset{r_2^*}{\rightleftarrows}} \begin{pmatrix}\text{bundles over} \\ \mathcal{A}_0(B_2)\end{pmatrix}$$

adjoint of r_2^* in suitable sense

(

We can write a formula for

$$\begin{pmatrix}\text{gauge-equivariant}\\ \text{bundles over } \mathcal{A}_0(B_1)\end{pmatrix} \xrightarrow{r_{2*}r_1^* \;=:\; Z(\Sigma)} \begin{pmatrix}\text{gauge-equivariant}\\ \text{bundles over } \mathcal{A}_0(B_2)\end{pmatrix}$$

which is again a path (direct) integral:

$$\langle \mathcal{K} | Z(\Sigma) | \mathcal{H} \rangle = \bigoplus\!\!\!\!\!\!\int_{\mathcal{A}(\Sigma)/\mathcal{G}(\Sigma)} \mathcal{D}A \;\; \mathcal{K}^{\dagger}_{A|_{B_2}} \otimes T_A \otimes \mathcal{H}_{A|_{B_1}}$$

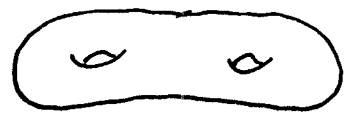
Hilbert-space-valued inner product of Hilbert Bundles

Direct Integral of Hilbert spaces

(analogous to $\langle \psi | Z(M) | \phi \rangle = \int \mathcal{D}A \; \overline{\psi(A|_{\Sigma'})} \, e^{iS[A]} \, \phi(A|_{\Sigma})$)

In the case where the $(n-1)$-mfld Σ is closed:

$$\emptyset \longrightarrow \Sigma \longleftarrow \emptyset$$



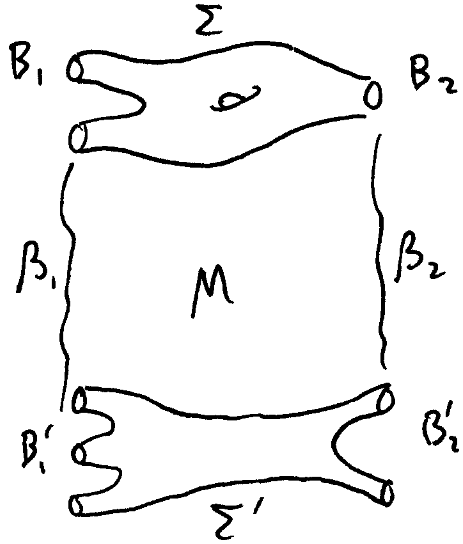
$\mathcal{A}(\emptyset) = \{A\}$, $\mathcal{G} = \langle e \rangle$ (one conn. & one gauge trans)

So an equivariant Hilbert bundle over $\mathcal{A}(\emptyset)$ is just a Hilbert space. It is "irreducible" iff this Hilbert space is 1-dim, i.e. $\cong \mathbb{C}$.

$$\langle \mathbb{C} | Z(\Sigma) | \mathbb{C} \rangle = \int_{\{A\}} \mathcal{D}A \;\; \mathbb{C} \otimes T_A \otimes \mathbb{C} \;\cong\; T$$

a Hilbert space.

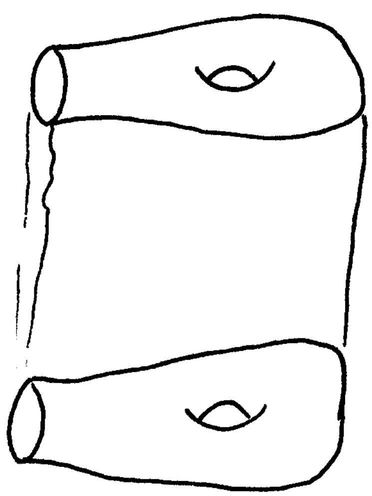
- Morton does something analogous to this to get "time evolution" for an n-mfld w/ corners



But we'll skip the details...

3d Quantum Gravity (with matter .... but not here!)

3d gravity is a TQFT, but only where $T_{\mu\nu} = 0$.
Suppose $T_{\mu\nu} \neq 0$ only in some bounded region.
Excise that region and do $\underline{\underline{e}}$TQFT on the resulting spacetime with boundary:



$$\begin{tikzcd}
S^1 \arrow[r] \arrow[d] & \Sigma \arrow[d] & \varnothing \arrow[l] \\
N \arrow[r] & M & \varnothing \arrow[l] \\
S^1 \arrow[r] \arrow[u] & \Sigma \arrow[u] & \varnothing \arrow[l]
\end{tikzcd}$$

For S^1:

- discretize :

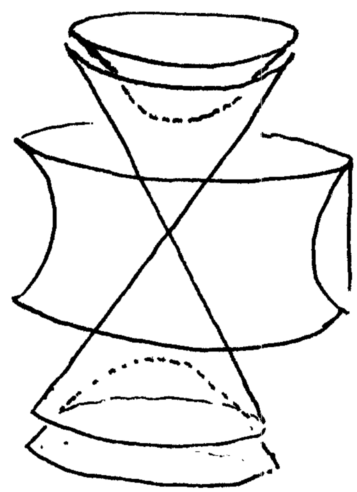
- describe connections by holonomies: $\mathcal{A}_0(S^1) = G$
- gauge transformations $\mathcal{G}(S^1) = G$ act by conjugation.

$\Rightarrow$ • $Z(S^1) = \{$gauge-equivariant bundles over connections$\}$

$\simeq \{AD(G)$-equivariant Hilbert bundles over $G\}$

For 3d gravity, $G = \begin{cases} SU(2) & \text{Riemannian} \\ SL(2,\mathbb{R}) & \text{Lorentzian} \end{cases}$

Focus on $SL(2,\mathbb{R})$. In this case the $AD(G)$ orbits in G near $1 \in G$ look like mass shells:



For an "irreducible" $AD(G)$-equivariant bundle, the Hilbert spaces will be trivial except over a single orbit (conjugacy class)

An equivariant bundle over an orbit is classified (up to iso.) by a representation of the stabilizer of a point in the orbit.

For 

, $\mathrm{Stab}(p) \cong U(1)$

The bundle is irreducible iff the $U(1)$ rep is,
So irred. equiv. bundles are labeled by ("mass shell", spin $\in \mathbb{Z}$).

What we're working on...

- Doing actual physics (or at least pseudophysics) with this stuff.

  3d Quantum gravity with boundaries

For this we must:

- Generalize the construction from finite G (Morton's work) to a Lie group (so far, compact Lie gp.)
  - Proper use of the action
    - action's role in finite case is invisible
    - need careful accounting ~~of~~ bdry terms.
    - examples
  - Mathematical issues:
    - Morton uses finiteness at some key points. (we think we can get around this, but it's more work!)
    - Nonfiniteness introduces new complications
      - continuity & smoothness
      - direct integral formulas require measures that transform appropriately

etc.

Key observation:

The gauge-equivariant bundles of Hilbert spaces can be viewed as representations of the von Neumann algebra associated to the action of gauge transformations on connections.

This leads to a different but equivalent (we think!) picture of eTQFT

$(n-2)$-mflds	$\sim$	von Neumann algebras
$(n-1)$-mflds	$\sim$	bimodules for the boundary algebras (Hilb. sp. that are l/r reps of the algs)
	$\sim$	bimodule homomorphisms ("time evolution of Hilbert spaces, with compatible time evolution of the boundary algebras.")