Topologically Massive Gravity with a Cosmological Constant

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Details and references at arXiv:0803.3998 [hep-th]
(or for the short story, 0807.0486)

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Topologically Massive Gravity

The "topologically massive" 2+1 gravity model has action

$$I[g] = \int d^3x \Big\{ \underbrace{-\sqrt{-g}\,R}_{\text{Einstein}} + \frac{1}{2\mu}\,\underbrace{\varepsilon^{\mu\nu\rho}\Big(\Gamma^{\alpha}_{\mu\beta}\partial_\nu\Gamma^{\beta}_{\rho\alpha} + \frac{2}{3}\,\Gamma^{\alpha}_{\mu\gamma}\Gamma^{\gamma}_{\nu\beta}\Gamma^{\beta}_{\rho\alpha}\Big)}_{\text{Chern–Simons}} \Big\}$$

giving field equations:

$$G_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = 0$$

where $C_{\mu\nu}$ is the Cotton tensor—the *conformal curvature* in 3d.

$$\begin{aligned} \mu^2 \to \infty \quad &\Longrightarrow \quad \text{pure 2+1 GR: flat geometry} \\ \mu^2 \to 0 \quad &\Longrightarrow \quad \text{conformally flat geometry} \end{aligned}$$

But for $0 < \mu^2 < \infty$ there are local propagating modes!
Linearized analysis: field redefinitions give "massive gravitons" obeying $(\Box + \mu^2)\varphi = 0$. The nonstandard Einstein–Hilbert sign is required for positive energy.

Adding a Cosmological Constant

The simple addition of a cosmological constant:

$$I[g] = \int d^3x \Big\{ -\sqrt{-g}\,(R - 2\Lambda) + \frac{1}{2\mu}\,\varepsilon^{\mu\nu\rho}\Big(\Gamma^{\alpha}_{\mu\beta}\partial_\nu\Gamma^{\beta}_{\rho\alpha} + \frac{2}{3}\,\Gamma^{\alpha}_{\mu\gamma}\Gamma^{\gamma}_{\nu\beta}\Gamma^{\beta}_{\rho\alpha}\Big)\Big\}$$

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = 0$$

has some surprising effects!

I'll consider only $\Lambda < 0$, and work mostly in units

$$\Lambda = -1$$

for simplicity.

To describe these effects, we first need a review of:

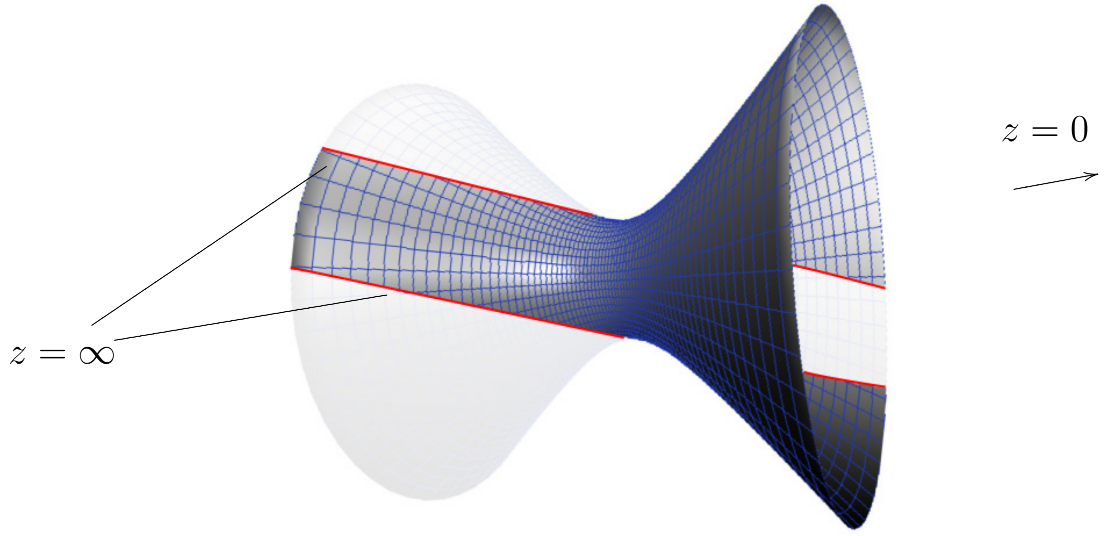
- Light-front coordinates in AdS
- Scalar fields in AdS

Light-front method

We work in a Poincaré patch in AdS$_3$, with two null coordinates x^+, x^- ($x^\pm := x \pm t$) and spacelike coordinate z, and metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{2dx^+ dx^- + dz^2}{z^2}$$

At $t = 0$, space looks like:



But we often view $x^+ \equiv \tau$ as “time” and (x^-, z) as “space”.
This setup is useful for studying fields in AdS backgrounds, e.g. …

Scalar fields in AdS$_3$

The action for a massive scalar field in AdS$_3$ background ($\Lambda = -1$)

$$I = -\frac{1}{2}\int d^3x\sqrt{-g}\left\{\partial_\mu\varphi g^{\mu\nu}\partial_\nu\varphi + m^2\varphi^2\right\},$$

written in light-front coordinates, *automatically* takes Hamiltonian form:

$$I = \int d\tau\left\{\langle\phi,\dot{\phi}\rangle - \left(\frac{1}{2}[\partial_z\phi]^2 + \frac{1}{2z^2}\left[m^2+\frac{3}{4}\right]\phi^2\right)dx^-dz\right\} \qquad \phi \equiv \frac{1}{\sqrt{z}}\varphi$$

where

$$\langle A, B\rangle \equiv \int dx^-dz\, A\,\partial_- B = -\langle B, A\rangle\,.$$

is the light-front symplectic structure. The field equations are:

$$\left[2\partial_-\partial_+ + \partial_z^2 - \frac{m^2+3/4}{z^2}\right]\phi = 0$$

Or, after a Fourier transform in $x^\pm$:

$$\left[\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} + \omega^2 - \frac{\nu^2}{z^2}\right]\left(\frac{\varphi}{z}\right) = 0 \qquad \text{Bessel eq. w. } \nu^2 = m^2+1$$

For

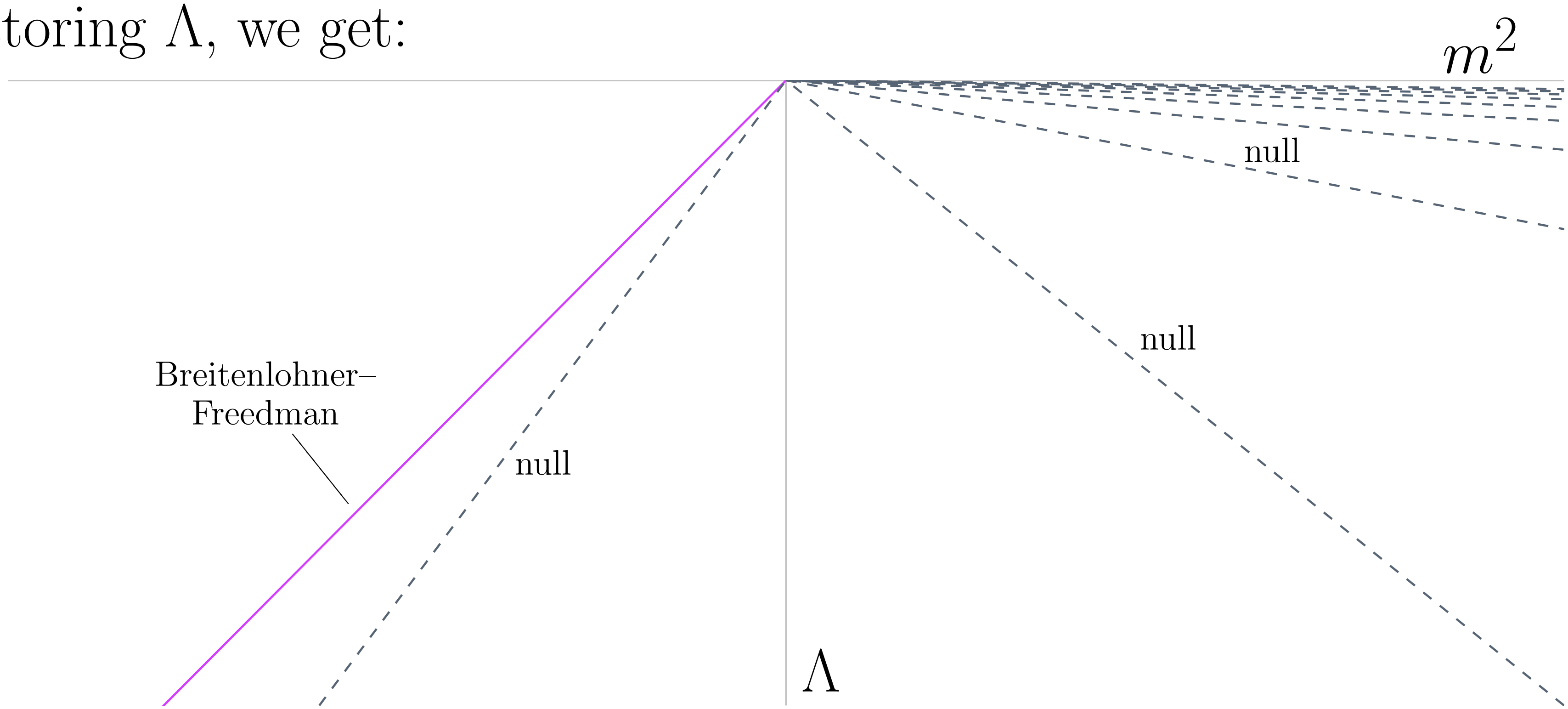
$$\nu^2 = m^2 + 1 = (1/2)^2\,,\ (3/2)^2\,,\ (5/2)^2\,,\ldots$$

the Bessel solutions reduce to:

$$\varphi \sim \Big(\text{slowly varying function of } z\Big)\cdot \exp(ik_\mu x^\mu)\,,$$

which exhibit *null* propagation. Also, the Breitenlohner–Freedman bound says we get a consistent field theory down to $m^2 = -1$.

Restoring Λ, we get:



Linearized Topologically Massive Gravity

The full action for topologically massive gravity is

$$I = \int d^3x \Big\{ -\sqrt{-\gamma}\,(R - 2\Lambda) + \frac{1}{2\mu}\,\varepsilon^{\mu\nu\rho}\Big(\Gamma^{\alpha}_{\mu\beta}\partial_\nu\Gamma^{\beta}_{\rho\alpha} + \frac{2}{3}\,\Gamma^{\alpha}_{\mu\gamma}\Gamma^{\gamma}_{\nu\beta}\Gamma^{\beta}_{\rho\alpha}\Big)\Big\}.$$

Any 1st order AdS metric fluctuation is equivalent, by a linearized diffeomorphism, to one in ‘light-front gauge’ :

$$ds^2 = \frac{2dx^+dx^- + dz^2}{z^2} + h_{++}\,(dx^+)^2 + 2h_+\,dx^+dz \; + h\,dz^2.$$

Up to quadratic order in the fluctuations h, h_+, h_{++}, the action is:

$$I = \int d\tau \left\{ -\frac{z^4}{\mu}\langle X, \dot{h}\rangle - \frac{z}{2}\Big([zX - h]^2 + \frac{z^3}{\mu}\Big[X + \partial h \; + \frac{\mu+2}{z}\,h\Big]\;Y\Big)dx^-dz \right\}$$

$$\text{with} \quad X \equiv \partial_- h_+\,, \qquad Y \equiv \partial_-^2\, h_{++}$$

The field equations for X and Y are algebraic; integrating them out and setting $\phi = z^{3/2}h$, we recover the standard AdS scalar action, with mass

$$m^2 = (\mu + 2)^2 - 1\,.$$

Chiral Mass Splitting

In fact, the components of the linearized Einstein tensor:

$$\mathcal{H}_{\rho\sigma} = \big[G_{\rho\sigma} - g_{\rho\sigma}\big]_{\rm LINEAR}$$

obey the AdS massive wave equation

$$\Big[2\partial_-\partial_+ + \partial_z^2 - \frac{m^2 + 3/4}{z^2}\Big]\big(\frac{1}{\sqrt{z}}\mathcal{H}_{\rho\sigma}\big) = 0$$

with mass depending on the mode's chirality:

Field	m^2
$\mathcal{H}_{++}$	$(\mu-2)^2-1$
$\mathcal{H}_{+z}$	$(\mu-1)^2-1$
$\mathcal{H}_{zz}\,,\,\mathcal{H}_{+-}$	μ^2-1
$\mathcal{H}_{-z}$	$(\mu+1)^2-1$
$\mathcal{H}_{--}$	$(\mu+2)^2-1$

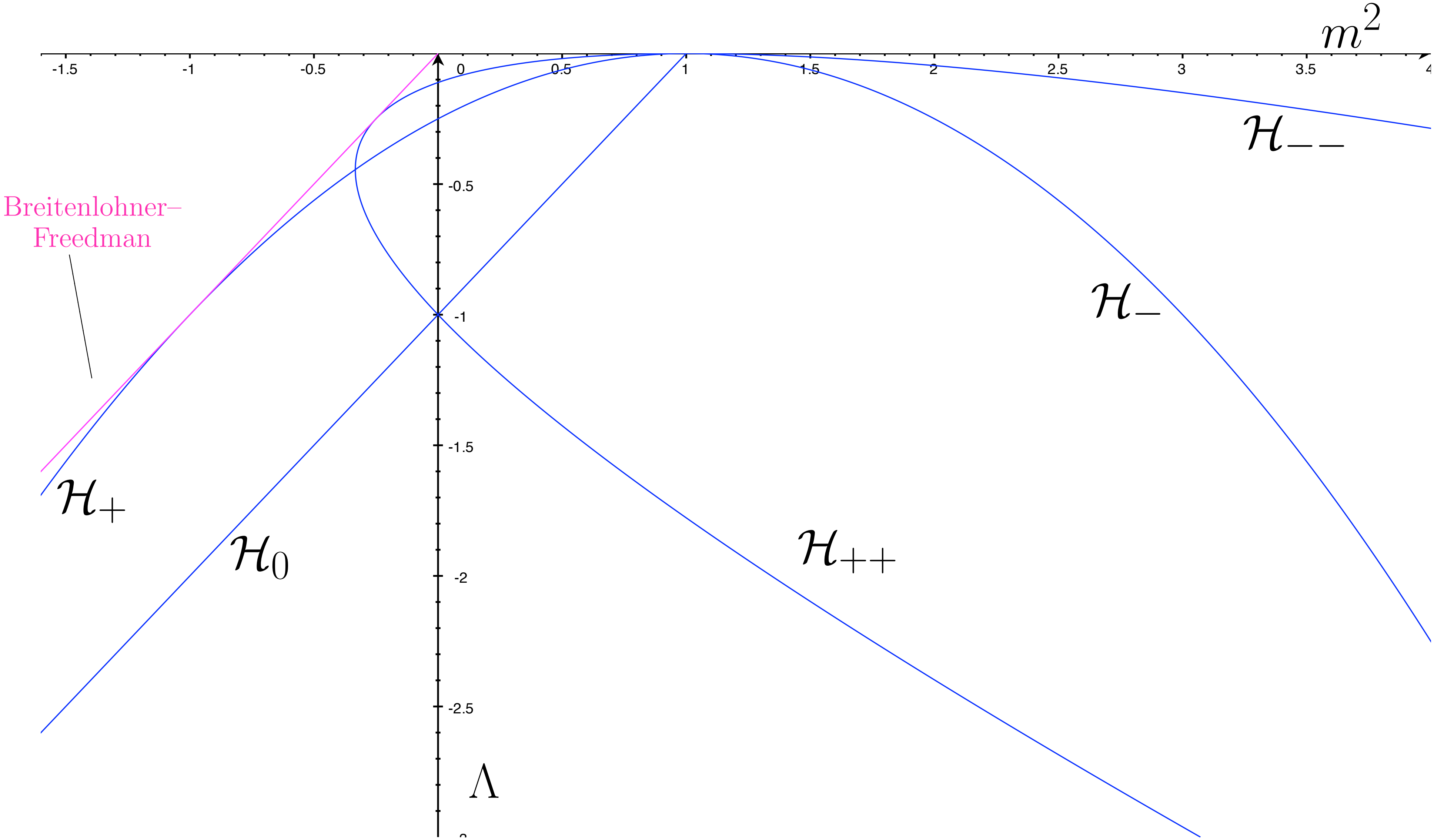
Or, restoring Λ:

Field	m^2
$\mathcal{H}_{++}$	$(\mu - 2\sqrt{-\Lambda})^2 + \Lambda$
$\mathcal{H}_{+z}$	$(\mu - \sqrt{-\Lambda})^2 + \Lambda$
$\mathcal{H}_{zz}\,,\,\mathcal{H}_{+-}$	$\mu^2 + \Lambda$
$\mathcal{H}_{-z}$	$(\mu + \sqrt{-\Lambda})^2 + \Lambda$
$\mathcal{H}_{--}$	$(\mu + 2\sqrt{-\Lambda})^2 + \Lambda$

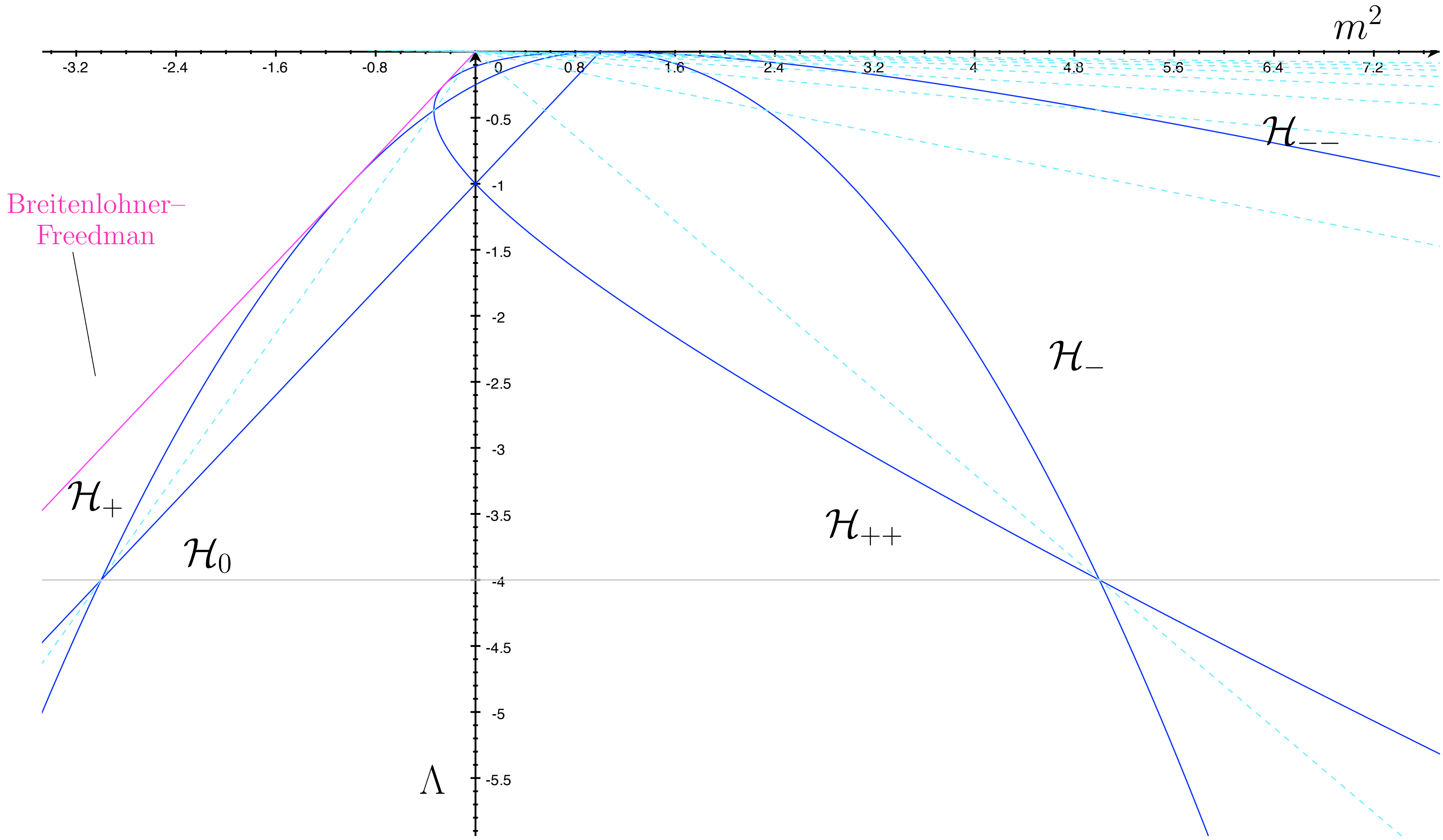
Notes:

1. Not independent components — all are derivable from the single "scalar" degree of freedom.
2. $m^2 \equiv \mu^2$ for $\Lambda = 0$.
3. invariance under simultaneous $\mu \mapsto -\mu$ and $+ \leftrightarrow -$.

Fixing μ (say $\mu = 1$) and adjusting Λ, the modes have these masses:



Or, zooming out a bit …



Asymptotics & Wave Packets

The basic solutions for linearized curvature fluctuations are Bessel functions, as we saw for AdS scalars. For these, we find:

- Solutions are asymptotically AdS near the $z = 0$ boundary;
- obey Fefferman-Graham asymptotics at $z = 0$ only if $\mu^2 > -\Lambda$.

However, the metric fluctuations *diverge* at $z = \infty$.

BUT, we can construct:

- A conserved symplectic current, yielding a Klein–Gordon inner product on linearized solutions.
- A *complete orthonormal system* of linearized solutions w.r.t. this inner product.

This lets us construct wave packets with arbitrary profile—in particular, support away from $z = \infty$.

Chern–Simons Formulation

The topologically massive gravity action (returning to $\Lambda = -1$) is:

$$I[g] = \int d^3x \Big\{ -\sqrt{-g}\,(R+2) + \frac{1}{2\mu}\,\varepsilon^{\mu\nu\rho}\Big(\Gamma^{\alpha}_{\mu\beta}\partial_\nu\Gamma^{\beta}_{\rho\alpha} + \frac{2}{3}\,\Gamma^{\alpha}_{\mu\gamma}\Gamma^{\gamma}_{\nu\beta}\Gamma^{\beta}_{\rho\alpha}\Big)\Big\}$$

Since pure gravity can be written as a Chern–Simons theory, and the second term *is* Chern–Simons, an obvious question is:

Can topologically massive gravity be given a pure Chern–Simons formulation?

Yes:

$$I_{\mathrm{TMG}}[e] = -\tfrac{1}{2}(1-\tfrac{1}{\mu})I_{\mathrm{CS}}[{}^{+}\!A[e]] + \tfrac{1}{2}(1+\tfrac{1}{\mu})I_{\mathrm{CS}}[{}^{-}\!A[e]]$$

with

$${}^{\pm}\!A_{\mu}{}^{a}{}_{b} = \omega_{\mu}{}^{a}{}_{b} \pm \varepsilon^{a}{}_{bc}e_{\mu}{}^{c}\,,$$

where e is the dreibein, $\omega = \omega[e]$ is the *torsionless* spin connection, and the coefficients $\frac{1}{2}(1 \pm \frac{1}{\mu})$ are the *central charges* of the boundary conformal field theory.

(Note: this theory isn't topological, because of the torsion constraint)

The critical value $\mu = 1$

When the topological mass takes the value $\mu = 1$, some interesting things happen:

- One of the coefficients in the Chern–Simons formulation disappears, leaving:

  $$I_{\mathrm{TMG}}[e] = I_{\mathrm{CS}}[{}^{-}A[e]]$$

- The *linearized* curvature fluctuations about AdS become "exact":

  $$\mathcal{H}_{\mu\nu} = D_{(\mu}F_{\nu)}\,,$$

  where D_μ is the covariant derivative *for the AdS background.* F_ν turns out obey precisely the equations for the dual field strength in "topologically massive electromagnetism":

  $$I = -\frac{1}{4}\int d^3x\left\{\sqrt{-g}\,F_{\mu\nu}g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma} + \mu\,\varepsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho\right\},$$

  So, we get an "equivalence" (at $\mu = 1$) between topologically massive spin-1 and spin-2 theories!

But, contrary to some recent claims, there are no jumps in degrees of freedom.

Some final remarks

Topologically massive gravity is a surprisingly rich toy model, with all of the essential conceptual difficulties of quantizing full-fledged GR:

- it is diffeomorphism invariant
- it has local propagating modes (unlike 3d GR)

These make quantization hard—for the same reasons QG is hard in general. So far, relatively little is known about quantizing the full nonlinear theory.

But, it's also simpler than 4d GR in some ways:

- 3d geometry conceptually easier than 4d geometry (less complicated curvature/fluctuations allowed)
- it has only *one* propagating degree of freedom

So, topologically massive gravity an interesting toy model which is less trivial than 3d Einstein gravity, but (perhaps) still simpler to quantize—using *your* favorite quantization programme—than full-fledged 4d gravity.

Again,

Details and references:	arXiv:0803.3998; or
The ‘short story’:	arXiv:0807.0486