Loops, Bubbles & Trace Divergences in Quantum Field Theory

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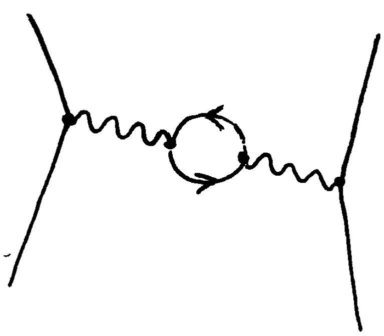

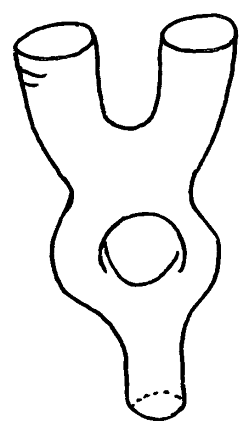

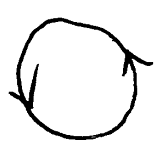

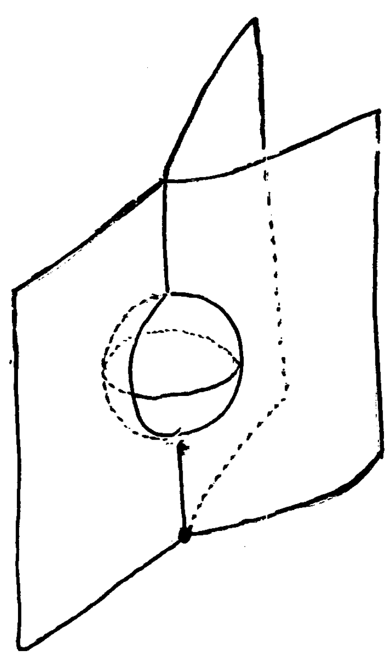

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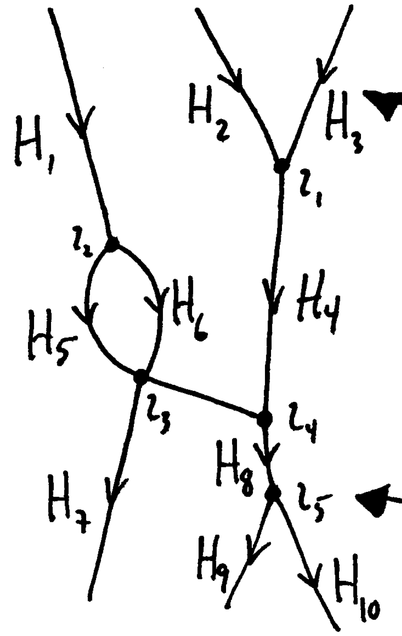
FEYNMAN DIAGRAMS



G ~ a Lie group

edges labelled by representations

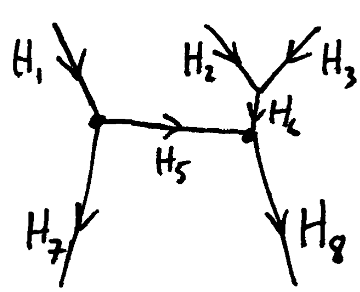
$$\rho_i : G \longrightarrow GL(H_i)$$

(often H is Hilb. sp. and the rep is unitary)

vertices labelled by intertwiners

$$\iota_5 : H_8 \longrightarrow H_9 \otimes H_{10}$$

To "evaluate" a Feynman diagram, we compose & tensor to get a single intertwiner (usu. summing over intermediate edge labels:



"evaluate" ⇝

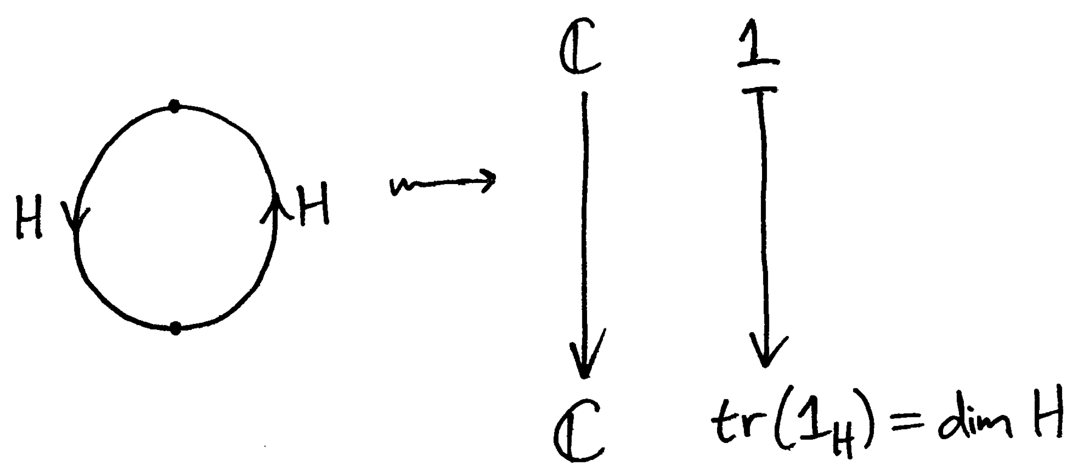
$$H_1 \otimes H_2 \otimes H_3$$
$$\downarrow$$
$$H_7 \otimes H_8$$

note: evaluation is unchanged by replacing

$H\uparrow$ by $\downarrow H^*$ so e.g.

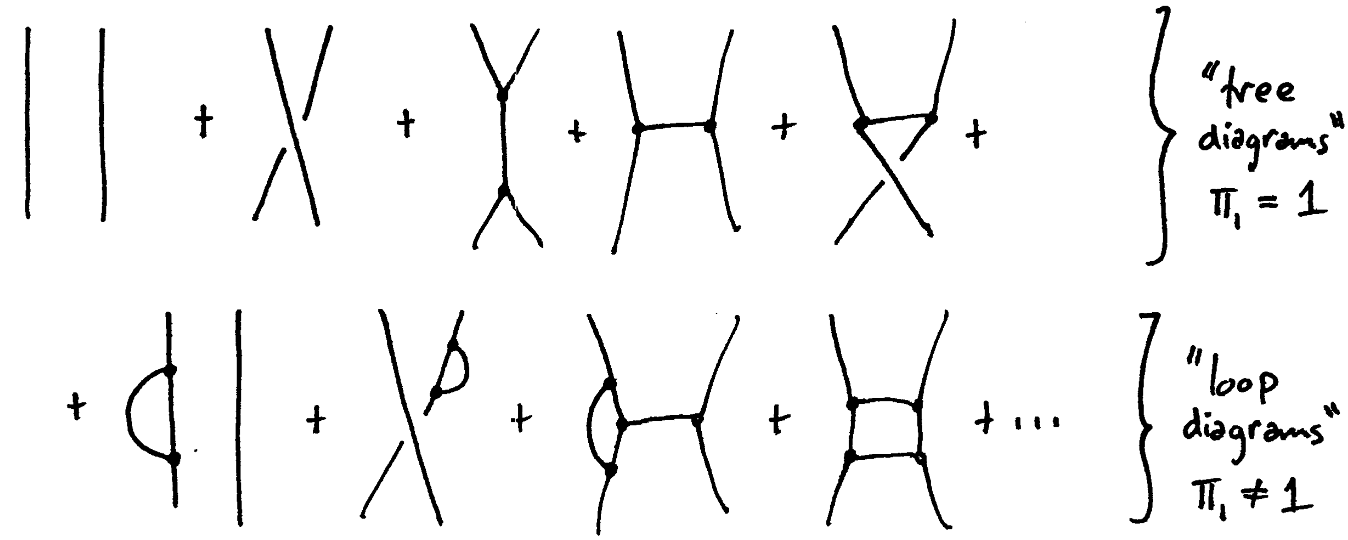
Problem: sometimes we don't get a well-defined intertwiner when we evaluate a diagram.

Simplest example:



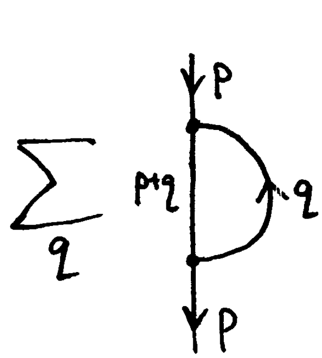
— a trace! This is only defined if H is finite dimensional.

QFT involves summing over Feynman diagrams with given input and output edges:



Tree diagrams are well defined: requiring the operators at vertices to be intertwiners puts a strong constraint on intermediate edge labels (physics jargon: <u>conservation laws</u> hold at the vertices)

Loop diagrams diverge because of "feedback" — i.e. certain (partial) traces diverge:

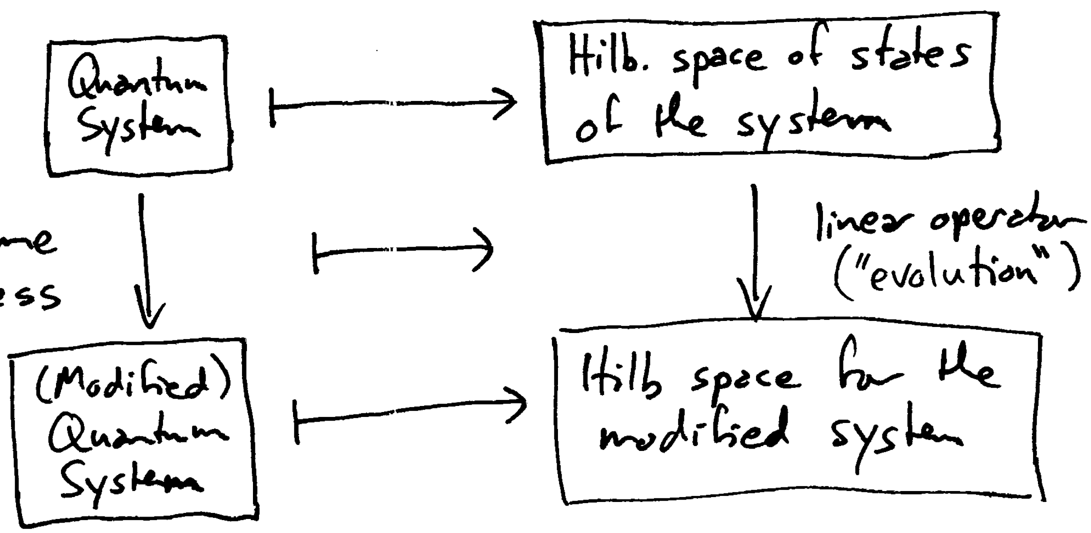


These kinds of divergences present a serious obstacle to describing real world QFT in a functorial way ...

FUNCTORIAL APPROACH TO QFT

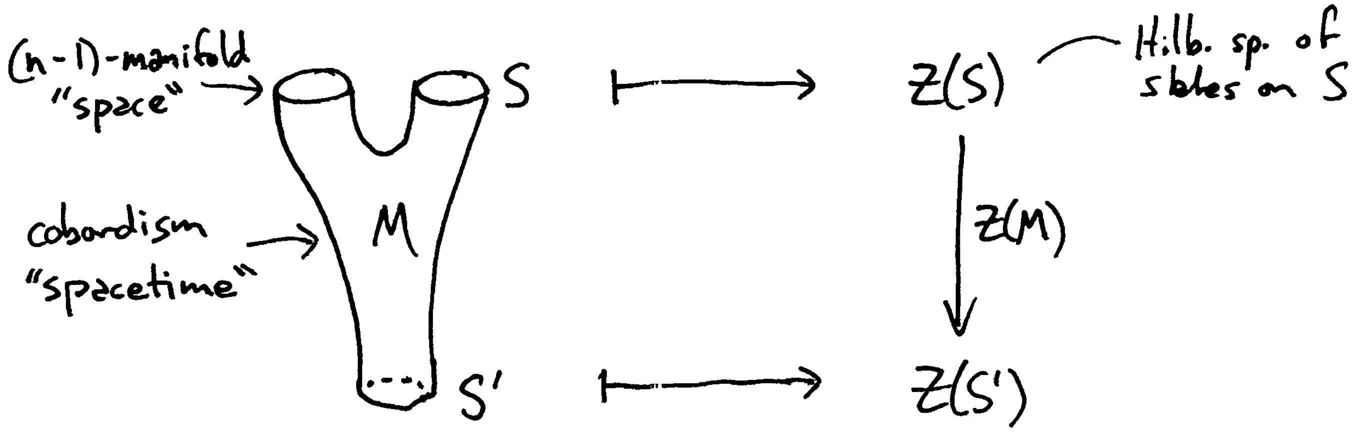
Basic idea:

$$(\text{"Quantum Systems"}, \text{Processes}) \longrightarrow (\text{Hilbert Spaces}, \text{Operators})$$

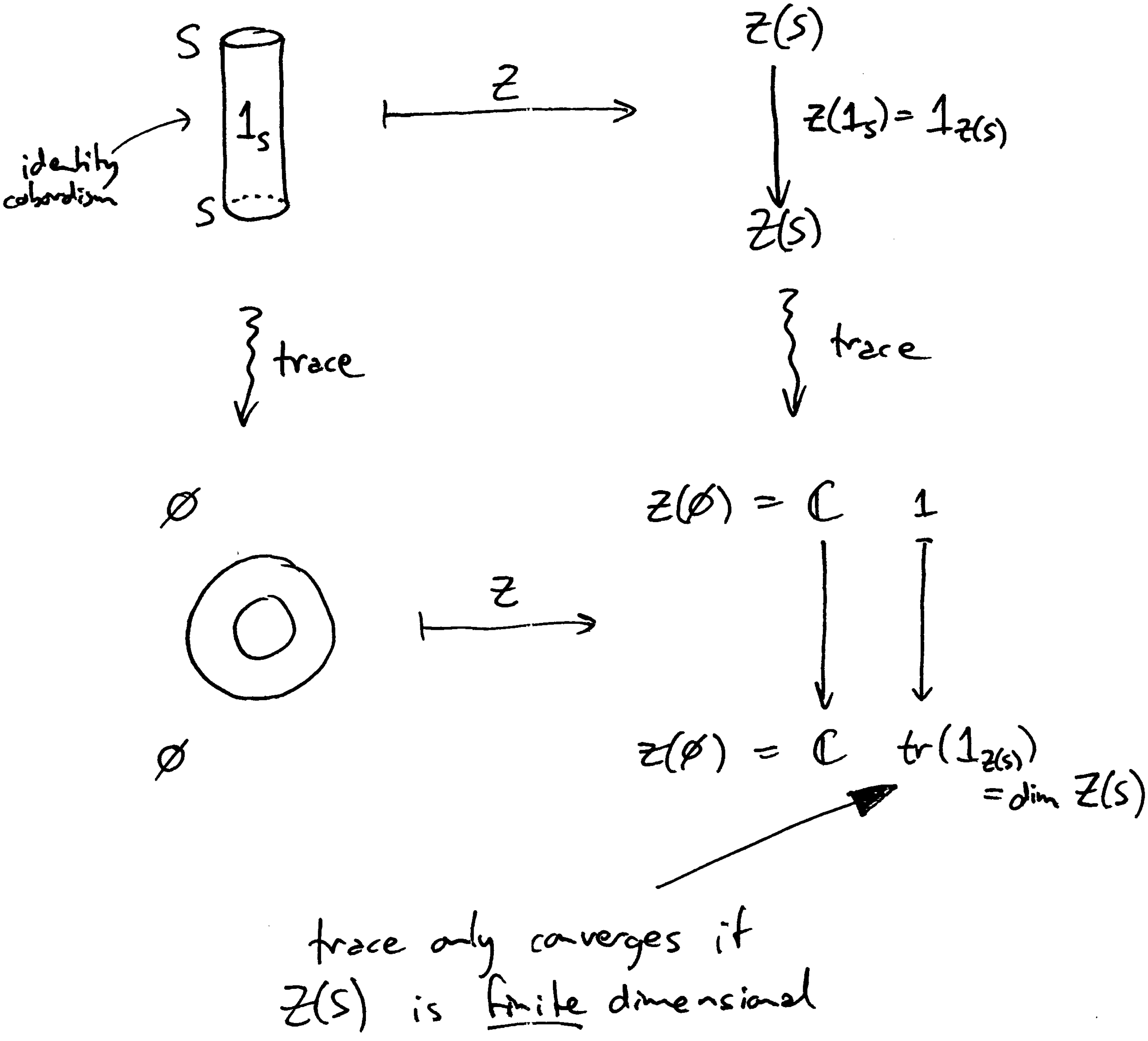


Main Example: Topological QFT (TQFT)

$$Z : n\mathrm{Cob} \longrightarrow \mathrm{Hilb}$$



"Problem" with TQFT: nCob is a "compact closed category" so it has <u>traces</u> (in Joyal-Street-Verity sense):

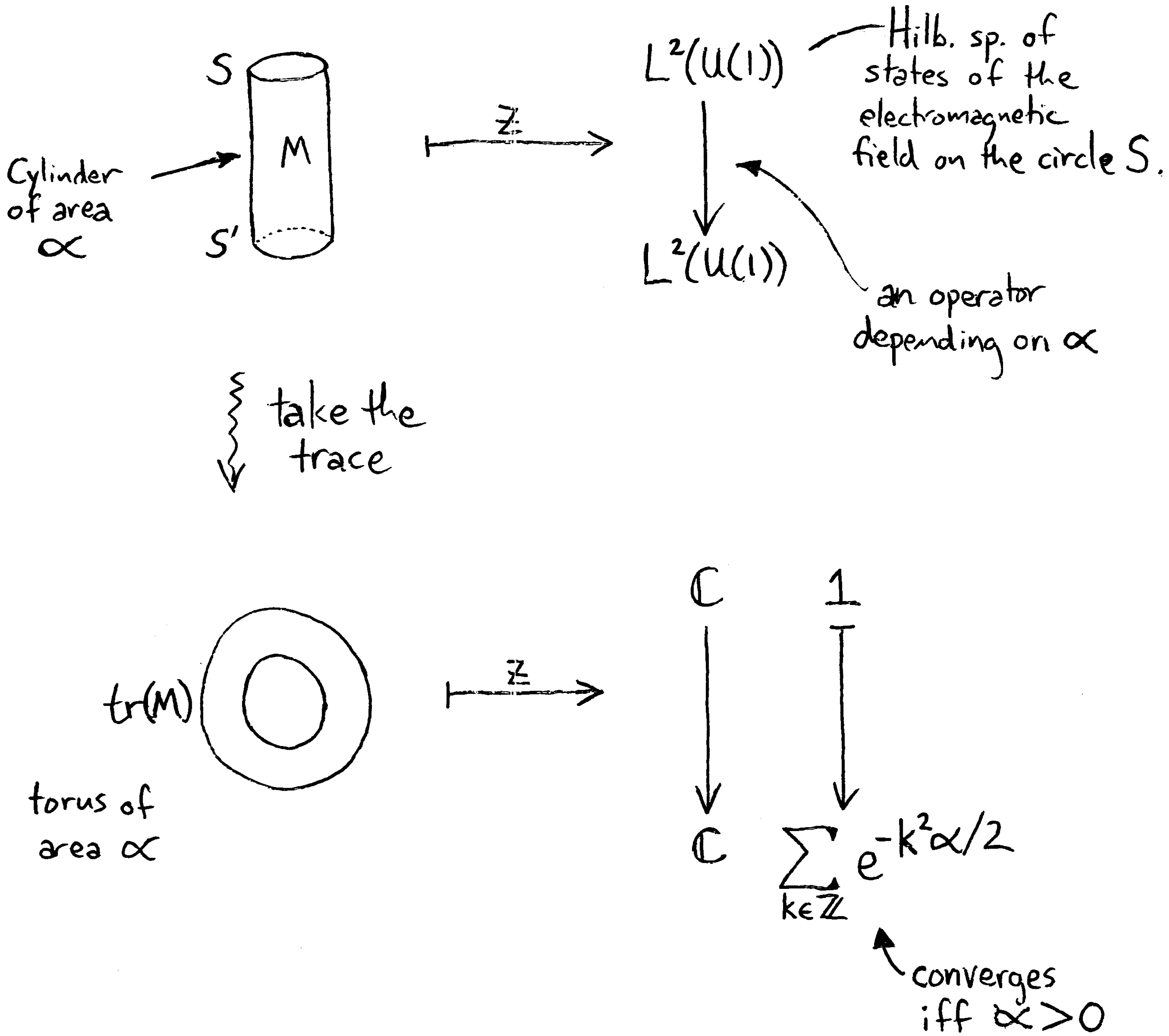


This is very unlike real physics, where Hilbert spaces of states are typically
L^2("Configuration space")

But there are theories that are "almost" TQFTs ....

2d Yang-Mills Theory

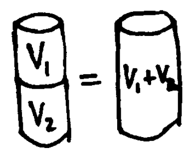
2d Yang-Mills theory is <u>almost</u> a TQFT: besides the topology of spacetime, it depends on the total <u>area</u>. For simplicity, consider 2d <u>electromagnetism</u> (2d YM with gauge group $U(1)$):



So, if we want to describe 2d electromagnetism as a functor

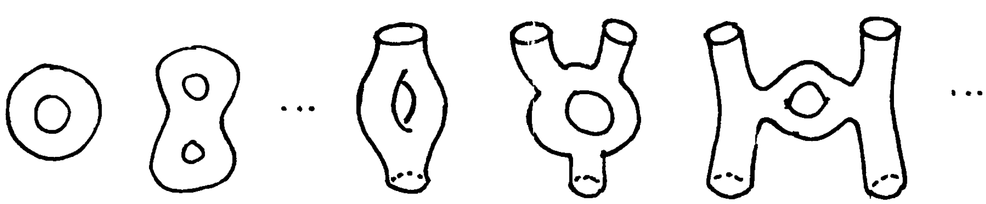
$$Z : \underline{\ ?\ } \longrightarrow \text{Hilb}$$

we must be careful with how we treat "cobordisms" of zero area, including:

- identity morphisms :  $V=0$
- braiding :  $V=0$
- in fact, <u>any</u> isomorphism (areas ≥ 0 & composition adds areas) 

there are various ways to deal with this....

But even worse, if we use a noncompact gauge group — $G = \mathbb{R}$ instead of $U(1)$ — then all of these spacetimes give divergences:



even when the area α is nonzero!
This is analogous to loop divergences in Feynman diagrams.

For electromagnetism with $G = \mathbb{R}$, there is in fact a simple topological condition for convergence: The theory converges <u>iff</u> the 1st cohomology of spacetime is trivial:

$$H^1 = 0$$

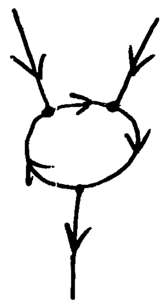
In particular, closed surfaces of genus ≥ 1 all give divergences!  ...

More generally, there is a theory called <u>p-form electromagnetism</u> (since it involves replacing the connection 1-form with a p-form) which:

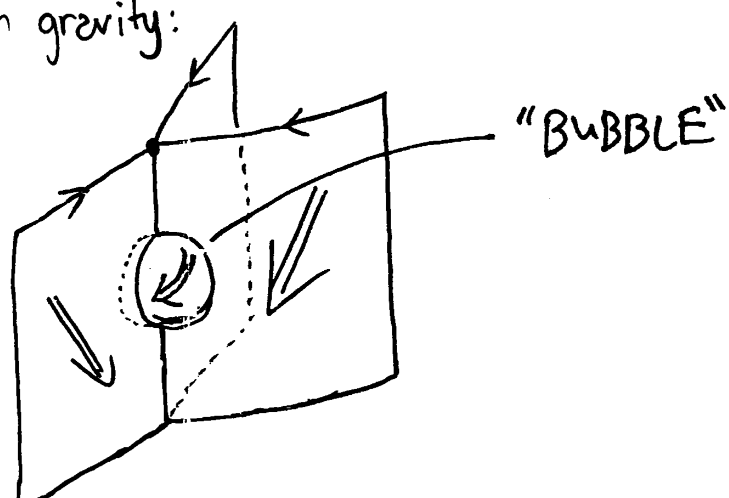
- in p+1 dimensions is almost a TQFT — only data besides topology is <u>volume</u>.
- for $G = \mathbb{R}$, converges iff

$$H^p = 0$$

The divergences (for $H^1 \neq 0$) in 2d electromagnetism are analogous to "loop divergences" in Feynman diagrams:



Divergences (for $H^2 \neq 0$) in 3d 2-form electromagnetism are analogous to "bubble divergences" in spin foams — higher dim. analogs of Feynman diagrams, used in quantum gravity:



But how are diagrams like this related to "TRACES"??

When we boost the dimension of our Feynman diagrams, we should really "boost the dimension" of the associated algebraic things as well (CATEGORIFICATION!)

Electromagnetism ⇝ 2-form electromagnetism

Groups ⇝ 2-groups

Vector spaces ⇝ 2-vector spaces

Hilbert spaces ⇝ 2-Hilbert spaces

Representations ⇝ 2-representations

⋮

etc.

Our divergences came from maps between Hilbert spaces, so we should ask what 2-Hilbert spaces are like...

Actually, let's just talk about 2-__Vector__ spaces...

VECTOR SPACES

Naively, a vector is a list of complex numbers

$$\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$$

So vector spaces are like

$$\mathbb{C}^n$$

↳ set of cx. #s

A map of vector spaces is a matrix of numbers:

e.g. $\mathbb{C}^2 \xrightarrow{T} \mathbb{C}^3$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \longmapsto \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \\ T_{31} & T_{32} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

matrix mult. w.r.t. +, × of complex numbers

2-VECTOR SPACES

Naively, a 2-vector is a list of complex vector spaces

$$\begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix}$$

So 2-vector spaces are like

$$\mathrm{Vect}^n$$

↳ category of cx. v.s.

A map of 2-vector spaces is a matrix of vector spaces

e.g. $\mathrm{Vect}^2 \xrightarrow{T} \mathrm{Vect}^3$

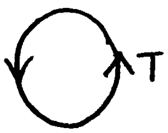
$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \longmapsto \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \\ T_{31} & T_{32} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

matrix mult. w.r.t $\oplus, \otimes$ of vector spaces

Maps like

$$\mathbb{C}^n \xrightarrow{T} \mathbb{C}^n$$

have traces:



$$\sum_i T^i{}_i \in \mathbb{C}$$

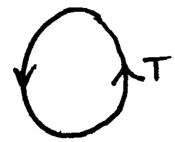
but these traces might diverge if we try switching to ∞-dim'l vector spaces.

(can't generally add up infinitely many complex numbers)

Maps like

$$\mathrm{Vect}^n \xrightarrow{T} \mathrm{Vect}^n$$

have traces



$$\bigoplus_i T^i{}_i \in \mathrm{Vect}$$

and this makes sense even if we try switching to ∞-dim'l 2-vector spaces, since we can ⊕ arbitrarily many vector spaces!

(need $\oint$ in 2-Hilbert space case)

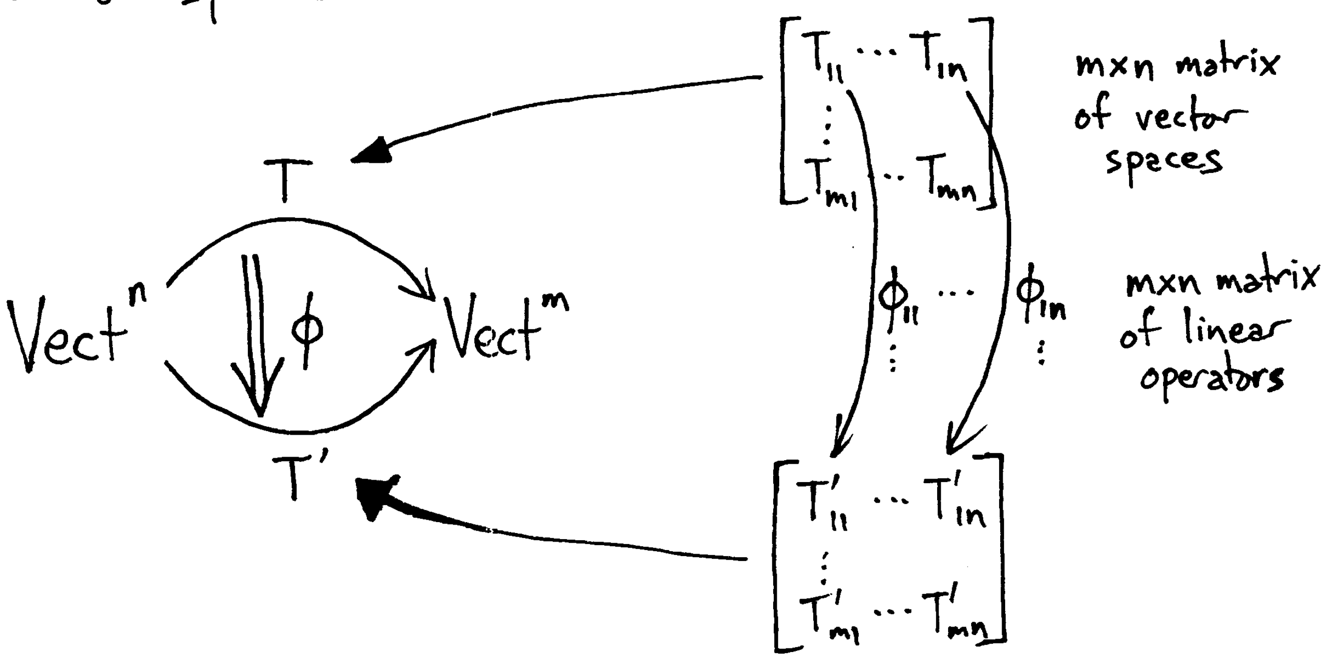
So: maps between 2-vector spaces are always "trace class"

BUT....

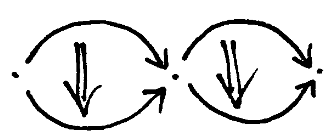
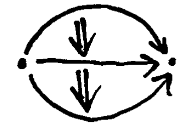
Besides maps between 2-vector spaces:

$$\mathrm{Vect}^n \xrightarrow{\;T\;} \mathrm{Vect}^m$$

there are 2-maps between maps between 2-vector spaces:

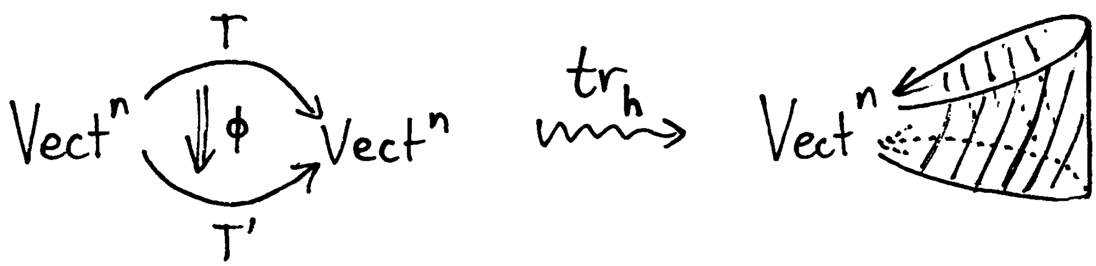


2-maps are just matrices of linear operators acting componentwise! They can be composed in two ways:

- horizontally: 
- vertically: 

These give rise to two distinct notions of trace...

Horizontal trace of a 2-map:



feed the horizontal "output"
into the horizontal "input".

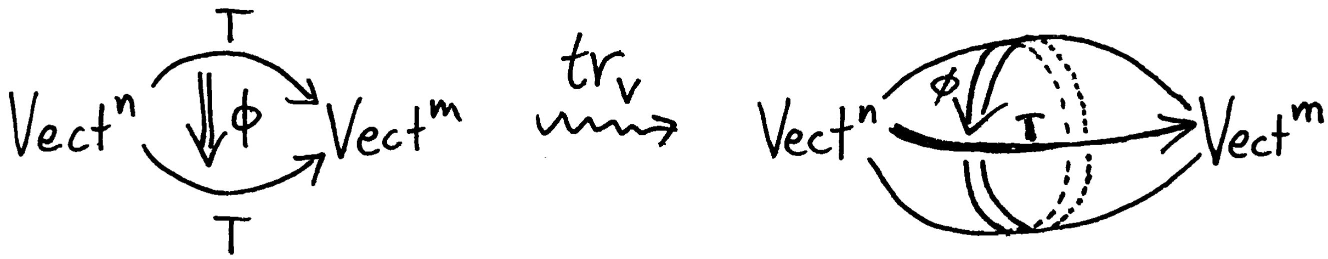
This gives

$$tr_h(\phi) : tr(T) \longrightarrow tr(T')$$

$$\text{i.e. } \bigoplus_i \phi^i{}_i : \bigoplus_i T^i{}_i \longrightarrow \bigoplus_i T^i{}_i$$

which makes perfect sense, even if we try using ∞-dimensional 2-Vector spaces

Vertical trace of a 2-map



feed the vertical "output"
back into the vertical "input"

This amounts to taking the trace of each of the linear operators in the matrix ϕ:

$$\mathrm{tr}_v(\phi) \in \mathrm{Mat}_{m \times n}(\mathbb{C})$$

w/ components $[\mathrm{tr}_v \phi]^i{}_j = \mathrm{tr}(\phi^i{}_j)$

But of course these traces will sometimes diverge unless the $T^i{}_j$ are always <u>finite-dimensional</u> vector spaces!

Moral: doing "physics" with 2-vector spaces (and 2-groups, 2-representations...) doesn't give "loop divergences" but BUBBLE divergences: