8) ⓐ $y'' - 4y' + 3y = 0$

Try $y = e^{rt}$. This gives $e^{rt}(r^2 - 4r + 3) = 0$

So $(r-3)(r-1) = 0 \Rightarrow r = 3$ or 1

and hence the general solution is

$$y = c_1 e^{3t} + c_2 e^{t}$$

ⓑ Letting $u = y'$, the diff. eq. can be written as $u' - 4u + 3y = 0$. Thus we have two differential equations in the variables u & y

$$y' = u$$
$$u' = -3y + 4u$$

or, as a matrix equation:

$$\begin{pmatrix} y \\ u \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix}$$

Eigenvalues are roots of $\det \begin{pmatrix} -\lambda & 1 \\ -3 & 4-\lambda \end{pmatrix} = \lambda^2 - 4\lambda + 3$
$= (\lambda - 3)(\lambda - 1)$

I.e., $\lambda = 3$ or 1.

Eigenvectors:

$\lambda = 3$: $\begin{pmatrix} -3 & 1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff 3v_1 = v_2$, so $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is an eigenvector.

$\lambda = 1$ $\begin{pmatrix} -1 & 1 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff v_1 = v_2$ so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a eigenvector.

general solution

$$\begin{pmatrix} y \\ u \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} = \begin{pmatrix} c_1 e^{3t} + c_2 e^{t} \\ 3c_1 e^{3t} + c_2 e^{t} \end{pmatrix}$$

Comparing to part ⓐ, we see that the first row here gives us the same solution for y (but, we also get $u = y'$ as the second row in part ⓑ).